RICEVIMENTO 11

Titolo nota 20/12/2007

Scritto d'Esame 2004.4 Problema 1

$$\lim_{n\to+\infty} \sqrt[n]{n^n+7^n+n^7}\cdot \arctan\left(\frac{2n}{n^2+3}\right) = 2$$

Brutale: $\sqrt[n]{n^n}\cdot \arctan\left(\frac{2\cancel{n}}{n^{\cancel{2}}}\right) = \sim \cancel{n}\cdot\frac{2}{\cancel{n}} = 2.$

Rigoroso

$$\sqrt[n]{n^n\left(1+\frac{7^n}{n^n}+\frac{n^7}{n^n}\right)}\qquad \frac{\arctan\left(\frac{2n}{n^2+3}\right)}{\frac{2n}{n^2+3}}\qquad \frac{2n}{n^2+3}$$

$$= \cancel{n}\cdot \underbrace{\boxed{\left(\quad\right)^{1/n}}}_{1^0}\ \underbrace{\boxed{\frac{\arctan(\ldots)}{(\ldots)}}}_{1}\ \frac{\cancel{n}}{\cancel{n^2}}\ \frac{2}{1+\frac{3}{n^2}} \to 2$$

$x = \frac{2n}{n^2+3}$

$\frac{3}{n^2}\to 0$

$$\lim_{x\to\infty} \frac{\arctan x}{x} \qquad \frac{\arctan(\text{Mostro})}{\text{Mostro}}$$

Pongo $x=$ Mostro. Quando $n\to+\infty$, Mostro $\to\infty$

Problema 2

$$f(x) = x^{700}\sin x^{300} - \log(1+x^{1000}) + \sinh x^{2004}$$

$f^{(n)}(0)\neq 0$ per quali n?

$$f(x) = x^{700}\left(x^{300} - \frac{1}{6}x^{900} + o(x^{900})\right) - x^{1000} + o(x^{1999})$$

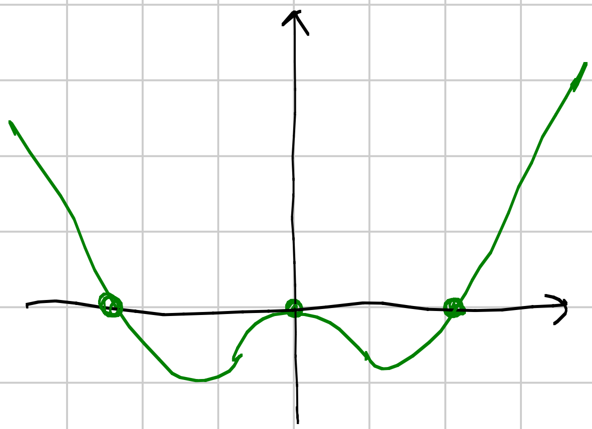
$$= \cancel{x^{1000}} - \frac{1}{6}x^{1600} + o(x^{1600}) - \cancel{x^{1000}}$$

$$= -\frac{1}{6}x^{1600} + o(x^{1600})$$

la prima derivata non nulla in $x=0$ è la 1600-esima

Quanto vale?

$$\frac{f^{(1600)}(0)}{1600!} x^{1600} = -\frac{1}{6} x^{1600}$$



$$f^{(1600)}(0) = -\frac{1600!}{6}.$$

L'eq. $f(x)=0$ ha almeno 3 soluzioni

* $f(0)=0$
* $x=0$ p.to di max. locale (per studio locale)
* $\lim_{x\to+\infty} f(x) = +\infty$ (per colpa di $\sinh x^{2004}$)

* f PARI

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Esercizio 3

$$\sum_{n=0}^{\infty} \underbrace{\frac{5^n+3}{n+1}}_{c_n} x^n$$ converge per quali x?

$$\lim_{n\to+\infty} \sqrt[n]{|c_n|} = \lim_{n\to+\infty} \sqrt[n]{\frac{5^n+3}{n+1}} = 5 = L \Rightarrow R = \frac{1}{5}$$

- $|x| < \frac{1}{5}$ $\rightsquigarrow$ converge
- $|x| > \frac{1}{5}$ $\rightsquigarrow$ Non converge
- $x = \frac{1}{5}$ $\rightsquigarrow \sum \frac{5^n+3}{n+1} \frac{1}{5^n}$ Diverge per C.A. con $b_n = \frac{1}{n}$

$$\lim_{n\to+\infty} \frac{a_n}{b_n} = \lim_{n\to+\infty} \frac{5^n+3}{n+1} \cdot \frac{1}{5^n} \cdot n = 1 \begin{array}{l} \neq 0 \\ \neq +\infty \end{array}$$

• $x = -\frac{1}{5}$ $\quad \sum \frac{5^n+3}{n+1}(-1)^n \frac{1}{5^n} =$

$$= \sum (-1)^n \left\{ \frac{5^n+3}{5^n} \cdot \frac{1}{n+1} \right\}$$

$$= \underbrace{\sum (-1)^n \frac{1}{n+1}}_{\text{converge per Leibnitz}} + \underbrace{\sum (-1)^n \frac{3}{5^n(n+1)}}_{\text{converge per Leibnitz o per assoluta convergenza.}}$$

$\Rightarrow$ La serie data converge $\Leftrightarrow$ $x \in \left[-\frac{1}{5}, \frac{1}{5}\right)$

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<u>Esercizio 4</u>

$$\begin{cases} u' = \frac{t}{\cos u} \\ u(0) = \alpha \end{cases}$$

$\frac{du}{dt} = \frac{t}{\cos u} \rightarrow \cos u \, du = t \, dt \rightarrow$

$\rightarrow \sin u = \frac{t^2}{2} + c$ $\qquad \alpha = \frac{\pi}{6}$

Sostituisco $t=0$ $\quad \sin u(0) = c$

$$\sin \frac{\pi}{6} = c \quad \longrightarrow \quad c = \frac{1}{2}$$

$$\sin u(t) = \frac{t^2}{2} + \frac{1}{2} = \frac{t^2+1}{2} \implies u(t) = \arcsin \frac{t^2+1}{2}$$

Per esistere deve essere

$$-1 \le \frac{t^2+1}{2} \le 1 \quad ; \quad -2 \le t^2+1 \le 2 \qquad t^2 \le 1 \to \underbrace{1 \le t \le 1}_{\text{INT. MAX. DI ESISTENZA}}$$

BANALE

Tempo di vita $T = 1$ $\qquad \lim_{T \to 1^-} u(t) = \arcsin 1 = \frac{\pi}{2}$

NO BLOW-UP

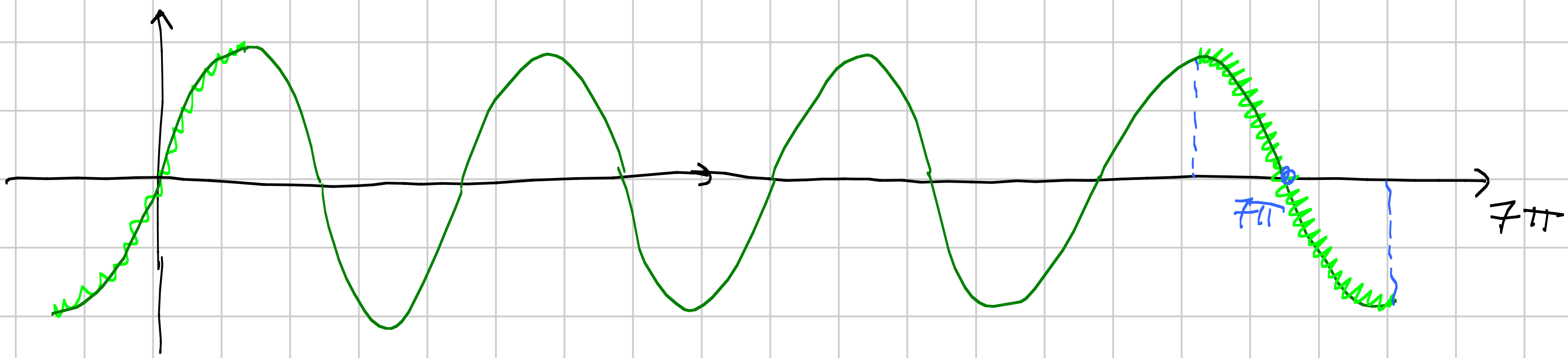
Facendo $u'(t)$ si vede che c'è BREAK-DOWN

$u(0) = 7\pi$ $\qquad \sin u = \frac{t^2}{2} + c$

$t = 0$ $\qquad \sin u(0) = c$

$\sin(7\pi) = c \implies c = 0$

$\sin u(t) = \frac{t^2}{2}$ $\qquad$ <u>verrebbe</u> da dire $u(t) = \arcsin \frac{t^2}{2}$



$7\pi - \arcsin \frac{t^2}{2} = u(t)$ $\qquad$ verifica $u(0) = 7\pi$

Sostituendo si vede che verifica anche l'equazione

$$\sum_{n=0}^{\infty} \underbrace{\frac{5^n+3}{n+1}}_{c_n} x^n = \sum_{n=0}^{\infty} \frac{5^n x^n}{n+1} + \sum_{n=0}^{\infty} \frac{3x^n}{n+1}$$

$$\sum_{n=0}^{\infty} \frac{(5x)^n}{n+1} \underset{\substack{\uparrow \\ y=5x}}{=} \sum_{n=0}^{\infty} \frac{y^n}{n+1} = 1 + \frac{y}{2} + \frac{y^2}{3} + \frac{y^3}{4} + \frac{y^4}{5} + \cdots$$

$$= \frac{1}{y}\left(y + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4} + \cdots\right)$$

$$= \frac{1}{y}\left(-\log(1-y)\right)$$

$$= \frac{1}{5x}\left(-\log(1-5x)\right)$$

$$3\sum_{n=0}^{\infty} \frac{x^n}{n+1} = 3\left(1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \cdots\right) = \frac{3}{x}\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right)$$

$$= \frac{3}{x}\left(-\log(1-x)\right).$$

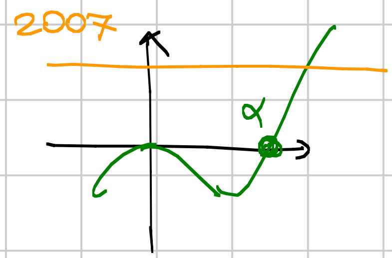
Prova d'esame 2007.2   Es. 2

$$f(x) = \sinh x^{2007} - \arctan x^{2006}$$

(a) $x=0$ stazionario di che tipo?

(b) $\exists\, \alpha>0$ t.c. $f(\alpha)=0$

(c) $f(x)=2007$ ha unica soluzione



(a) $f(x) = -x^{2006} + o(x^{2006}) \longrightarrow$ Max locale

(b) $\displaystyle\lim_{x\to+\infty} f(x) = +\infty$

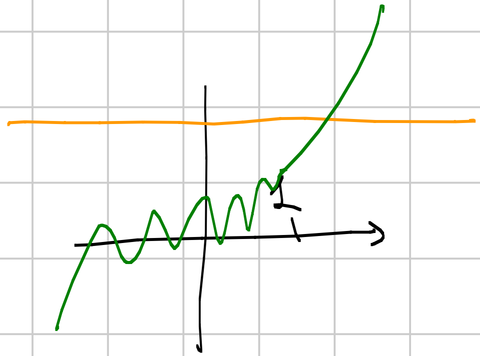
(c) $$f'(x) = \cosh x^{2007} \cdot 2007\, x^{2006} - \frac{2006 \cdot x^{2005}}{1+x^{2006\cdot 2}}$$

$$= x^{2005}\left\{ \underbrace{2007x \cdot \cosh x^{2007}}_{\geq 2007} - \underbrace{\frac{2006}{1+x^{4012}}}_{\leq 2006} \right\}$$

Per $x \geq 1$ si ha che $f'(x) > 0$

Per $x \geq 1$ $f(x)$ è crescente, quindi $f(x) = 2007$ ha al massimo una soluzione $x \geq 1$

Resta da dim. che non ci sono soluz. per $x \leq 1$, ma per $x \leq 1$ ho che



$$f(x) = \overbrace{\sinh x^{2007}}^{\text{CRESCENTE}} - \arctan x^{2006} \leq$$

$$\leq \sinh 1 < 2007$$

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Scritto d'esame 2004.6 / Esercizio 3

$$\begin{cases} a_{n+1} = \sqrt{a_n^2 - a_n + 7} \\ a_0 = 2005 \end{cases}$$

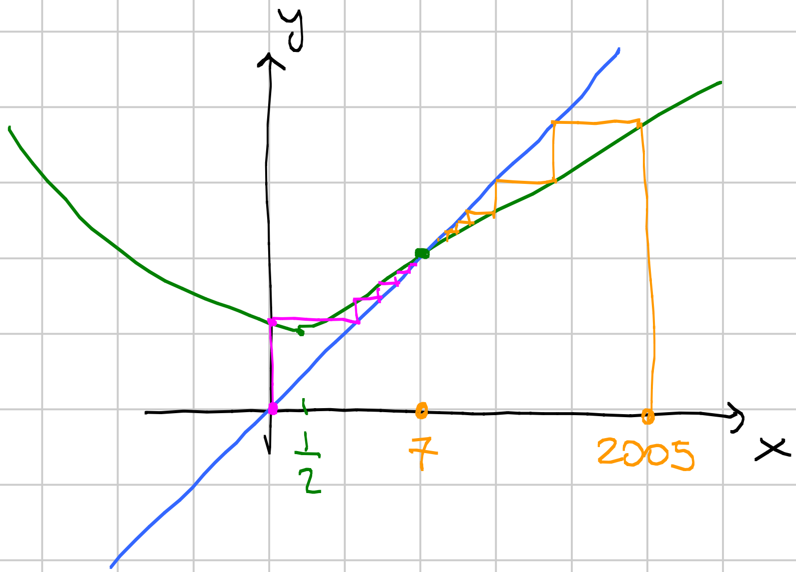
$$f(x) = \sqrt{x^2 - x + 7}$$

$$f(x) = x \qquad f(x) > x$$

$$\sqrt{x^2 - x + 7} \geq x \qquad \underbrace{\cancel{x^2} - x + 7 \geq \cancel{x^2} \qquad x \leq 7}_{\text{TROPPO BRUTALE}}$$

Per quali x ha senso la radice? $x^2-x+7 \geq 0$

$$x_{1,2} = \frac{1 \pm \sqrt{1-28}}{2}$$ radici complesse $\Rightarrow$ sempre verificata



$$f'(x) = \frac{2x-1}{2\sqrt{x^2-x+7}} > 0$$

$\Leftrightarrow$ $2x-1>0$ $\Leftrightarrow$ $x > \frac{1}{2}$

$\frac{1}{2}$

$\searrow$ $\nearrow$

FATTI IMPORTANTI SULLA $f(x)$

* $f(x) = x \Leftrightarrow x = 7$ (F1)
* $f(x) \leq x$ per $x \geq 7$ (F2)
* $f(x)$ crescente per $x \geq \frac{1}{2}$ (F3)

PIANO PER SUCCESSIONE

(i) $7 \leq a_n \quad \forall n \in \mathbb{N}$

(ii) $a_{n+1} \leq a_n \quad \forall n \in \mathbb{N}$

(iii) $a_n \to \ell \in \mathbb{R}$

(iv) $\ell = 7$

Dim (i) INDUZIONE $\boxed{n=0}$ $a_0 \overset{?}{\geq} 7$ $2005 \geq 7$ CERTO!

$\boxed{\text{P.I.}}$ Ipotesi: $a_n \geq 7$ Tesi $a_{n+1} \geq 7$

Dim $a_n \geq 7$ Applico f (che per F3 è crescente sopra $\frac{1}{2}$)

$$f(a_n) \geq f(7)$$
$$a_{n+1} \geq 7$$

Dim (ii) $a_{n+1} \overset{?}{\leq} a_n$ $f(a_n) \overset{?}{\leq} a_n$ Questo segue dal Fatto 2 e dal punto (i)

Dim (iii) a_n deb. decr. + limitata dal basso $\Rightarrow$ esiste limite reale

Dim. (iv)

$$\begin{array}{ccccc} a_{n+1} & = & f(a_n) & & \\ \downarrow & & \downarrow & \boxed{F1} & \\ \ell & = & f(\ell) & \Longleftrightarrow & \ell = 7. \end{array}$$

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Studiando la successione con $a_0 = 0$ conviene pensarla con $a_1 = \sqrt{7}$, così siamo già nella zona con $x \geq \frac{1}{2}$

(i) $\frac{1}{2} \leq a_n \leq 7$ $\boxed{\forall n \in \mathbb{N}}$ $\forall n \geq 1$

(ii) $a_{n+1} \geq a_n$ $\forall n \in \mathbb{N}$

(iii) $a_n \to \ell \in \mathbb{R}$ ← analogo

(iv) $\ell = 7$ ← come sopra

Dim (i): $n=1$ ok

$\boxed{\text{P.I.}}$ Ipotesi: $a_n \leq 7$ Tesi: $a_{n+1} \leq 7$

Dim. prendo l'ipotesi e applico f

$$a_n \leq 7 \to \underset{a_{n+1}}{\underset{\parallel}{f(a_n)}} \leq \underset{7}{\underset{\parallel}{f(7)}}$$

NO!!!!!

Nuovo p.to (i)

$$\frac{1}{2} \leq a_n \leq 7 \to f\left(\tfrac{1}{2}\right) \leq f(a_n) \leq f(7)$$

$$\frac{1}{2} \leq \sqrt{6,\ldots} \leq a_{n+1} \leq 7$$

Scritto 2005.2 Esercizio 2

$$\int_1^{+\infty} \frac{dx}{e^{2x}-1}$$ Unico problema $+\infty$

$f(x) = \frac{1}{e^{2x}-1}$ si comporta come $\frac{1}{e^{2x}} = e^{-2x}$

$\Rightarrow$ converge

$$= \lim_{A\to+\infty} \int_1^A \frac{dx}{e^{2x}-1}$$

$\int \frac{1}{e^{2x}-1} dx =$ Pongo $y = e^x$ $dy = e^x dx$

$$\int \frac{1}{e^x(e^{2x}-1)} \boxed{e^x dx}^{dy} = \int \frac{dy}{y(y^2-1)}$$ $\frac{1}{y(y^2-1)} = \frac{A}{y} + \frac{B}{y+1} + \frac{C}{y-1}$

$=$ viene

$$\int_0^{+\infty} \frac{dx}{e^{2x}-1} = \underbrace{\int_0^{3/2}}_{\text{DIVERGE}} + \underbrace{\int_{3/2}^{+\infty}}_{\text{converge}} = +\infty$$

Per $x \to 0$, $\dfrac{1}{e^{2x}-1} \sim \dfrac{1}{\cancel{1}+2x-\cancel{1}} = \dfrac{1}{2x} \Rightarrow$ Diverge

Rigoroso: C.A. con $g(x) = \dfrac{1}{x}$

$$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{1}{e^{2x}-1}\cdot x = \lim_{x\to 0^+} \frac{2x}{e^{2x}-1}\,\frac{1}{2} = \frac{1}{2} \begin{matrix}\neq 0\\ \neq +\infty\end{matrix}$$

$\Rightarrow$ Stesso comportamento.

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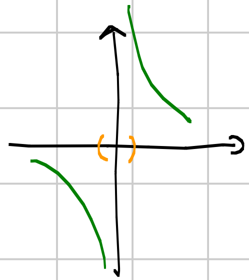
$$\int_{-1}^{1} \frac{1}{e^{2x}-1}\,dx = \int_{-1}^{0} + \int_{0}^{1}$$

DIV. A $-\infty$ DIV. A $+\infty$



$=$ INDETERMINATO

$\boxed{\text{PAG } 109}$

$$\sum_{n=0}^{\infty} \frac{n}{n+1} x^{2n} \underset{x^2=y}{=} \sum_{n=0}^{\infty} \frac{n}{n+1} y^n = \sum_{n=0}^{\infty} \frac{n+1-1}{n+1} y^n$$

$$= \sum_{n=0}^{\infty} \left(1 - \frac{1}{n+1}\right) y^n = \sum y^n - \sum \frac{y^n}{n+1}$$