RICEVIMENTO 10

Titolo nota 19/12/2007

$\boxed{\text{SCRITTO D'ESAME 2004.5}}$

$$\lim_{x\to 0} \frac{(1+x)^{x^2} - (1+x^2)^x}{\sin^2(x^2+x^3)}$$

Brutale $\sin^2(x^2+x^3) \sim \sin^2 x^2 \sim x^4$ sviluppiamo tutto di ordine 4

$$\sin^2(x^2+x^3) = x^4 + o(x^4)$$

$$(1+x)^{x^2} = e^{x^2\log(1+x)} = e^{x^2\left(x - \frac{x^2}{2} + o(x^2)\right)}$$

$$= e^{x^3 - \frac{x^4}{2} + o(x^4)} = 1 + x^3 - \frac{x^4}{2} + o(x^4)$$

$$(1+x^2)^x = e^{x\log(1+x^2)} = e^{x\left(x^2+o(x^3)\right)} = e^{x^3+o(x^4)} = 1+x^3+o(x^4)$$

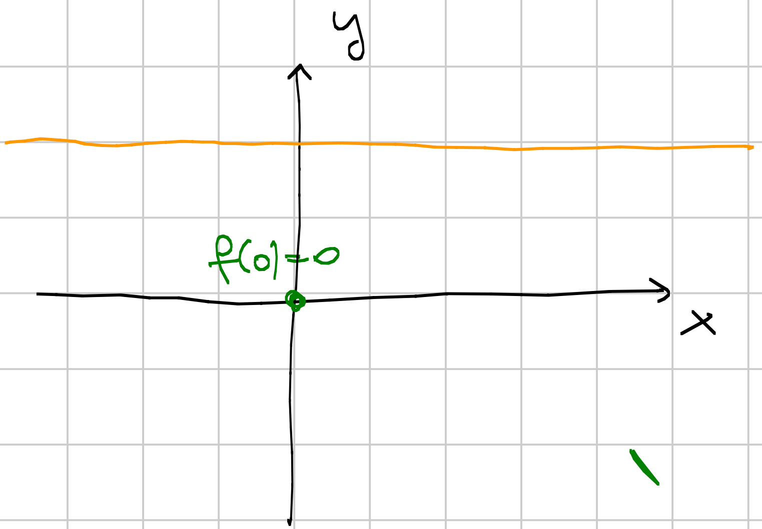
$$\frac{\text{Num}}{\text{Den}} = \frac{\cancel{1}+\cancel{x^3}-\frac{x^4}{2}-\cancel{1}-\cancel{x^3}+o(x^4)}{x^4+o(x^4)} =$$

$$= \frac{\cancel{x^4}\left(-\frac{1}{2}+\frac{o(x^4)}{x^4}\right)}{\cancel{x^4}\left(1+\frac{o(x^4)}{x^4}\right)} \to -\frac{1}{2}$$

—o—o—

② Trovare il + grande λ per cui

$$\underbrace{\lambda x - \frac{1}{3}x^3}_{f(x)} \leq 1 \qquad \forall x \geq 0$$



Mi serve che il max di $f(x)$ per $x \geq 0$ sia ≤ 1

$$f'(x) = \lambda - x^2$$

$f'(x) = 0 \iff x^2 = \lambda$

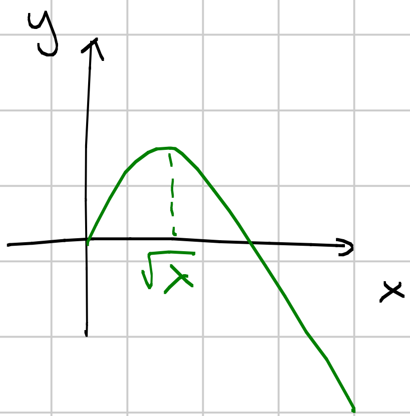
- se $\lambda < 0$, allora $f'(x) = \lambda - x^2 < 0$ sempre quindi

$f(x)$ decrescente sempre, quindi $f(x) \leq \underbrace{f(0)}_{0} \quad \forall x \geq 0$

- se $\lambda \geq 0$, allora $f'(x)$ si annulla per $x = \pm\sqrt{\lambda}$

A noi interessa $x = \sqrt{\lambda}$. Quanto vale $f(\sqrt{\lambda}) = \lambda\sqrt{\lambda} - \frac{\lambda\sqrt{\lambda}}{3}$

$$= \frac{2}{3}\lambda\sqrt{\lambda}$$

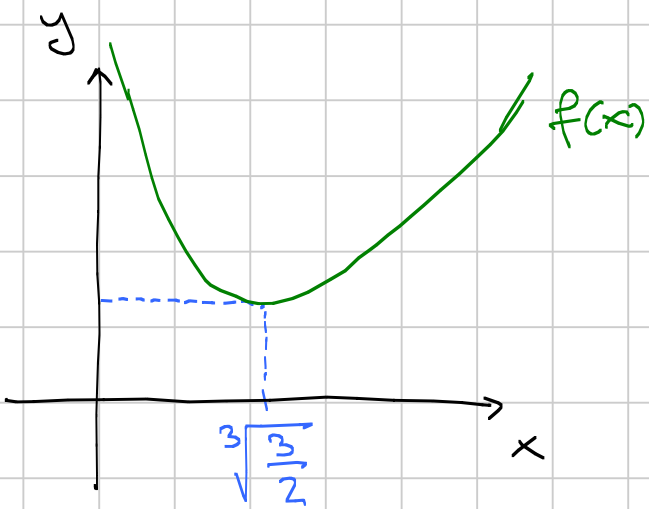


Impongo $\frac{2}{3}\lambda\sqrt{\lambda} \leq 1$

$\lambda^{3/2} \leq \frac{3}{2} \qquad \lambda \leq \left(\frac{3}{2}\right)^{2/3} = \sqrt[3]{\frac{9}{4}}$

Risoluzione alternativa

$$\lambda x - \frac{1}{3}x^3 \leq 1 \qquad \lambda x \leq 1 + \frac{1}{3}x^3$$



$$\lambda \leq \underbrace{\frac{1}{x} + \frac{1}{3}x^2}_{f(x)} \quad \text{per } x > 0$$

$$f'(x) = -\frac{1}{x^2} + \frac{2}{3}x$$

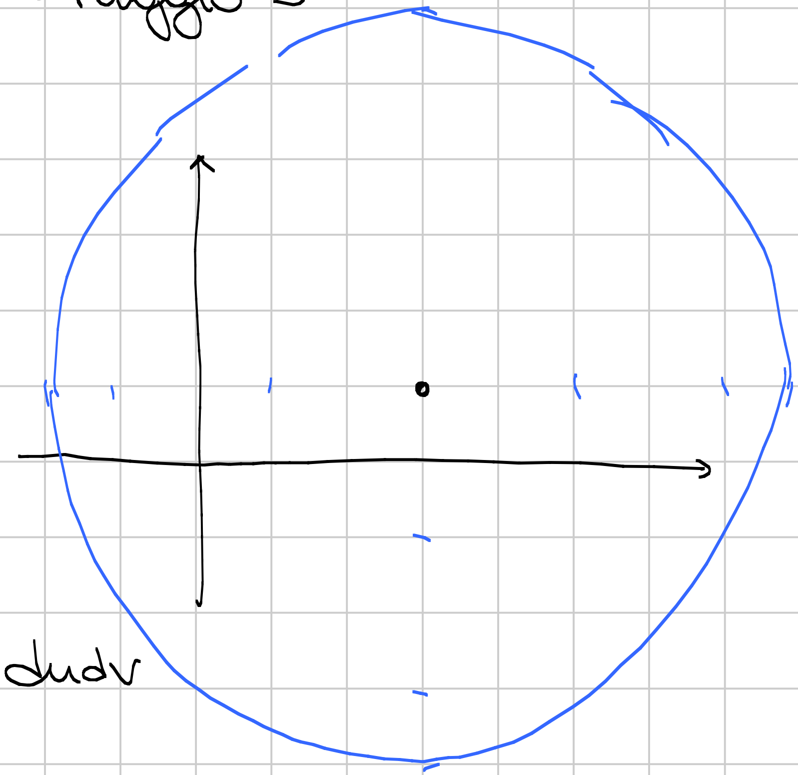
$$= \frac{-3+2x^3}{3x^2}$$

$$f'(x) > 0 \iff 2x^3 - 3 > 0 \iff x^3 > \frac{3}{2} \iff x > \sqrt[3]{\frac{3}{2}}$$

$$f\left(\sqrt[3]{\frac{3}{2}}\right) = \text{sostituire...}$$

④ $$\iint_C (x+2y)\,dx\,dy$$

C = cerchio con centro in (3,1) e raggio 5



$$\underbrace{(x-3)}_{=u}^2 + \underbrace{(y-1)}_{=v}^2 \le 25$$

$x-3=u \qquad x=u+3$

$y-1=v \qquad y=v+1$

$$\iint_C (x+2y)\,dx\,dy = \iint_{u^2+v^2\le 25} (u+3+2v+2)\,du\,dv$$

$$= \iint_{u^2+v^2\le 25} (u+2v+5)\,du\,dv = \underbrace{\iint u}_{=0} + 2\underbrace{\iint v}_{=0} + \iint 5$$

$$= 5 \text{ Area cerchio} = 5\cdot 25\pi = 125\pi$$

Su quali cerchi l'integrale fa 0 ?

Centro (x_0, y_0), raggio : R

$$\underbrace{(x-x_0)}_{u}{}^2 + \underbrace{(y-y_0)}_{v}{}^2 \le R^2$$

$x = u + x_0$

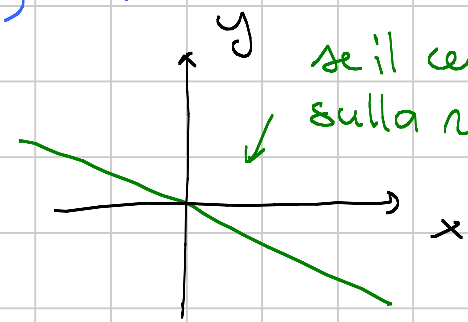
$y = v + y_0$

J=1 perché traslazione

$$\iint_C (x+2y)\,dx\,dy = \iint_{u^2+v^2\le R^2} (u+x_0+2v+2y_0)\,du\,dv$$

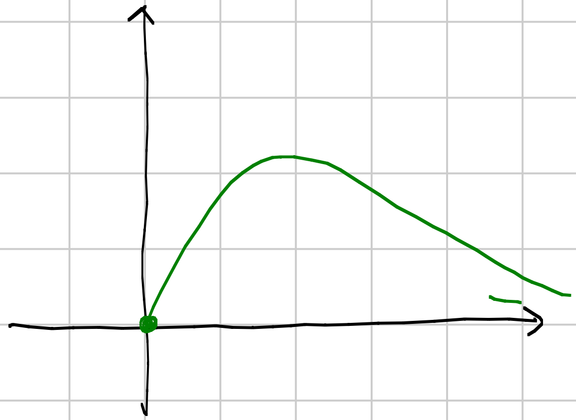
$$= \underbrace{\iint u}_{=0} + \underbrace{\iint 2v}_{=0} + \underbrace{\iint (x_0+2y_0)}_{=(x_0+2y_0)\pi R^2}$$

Quindi integrale $=0 \iff x_0+2y_0=0$



se il centro sta sulla retta int $=0$

③ $a_{n+1} = \underbrace{\frac{3a_n}{a_n^2+2}}_{f(x)}$ $\qquad a_0 = \frac{1}{2004}$



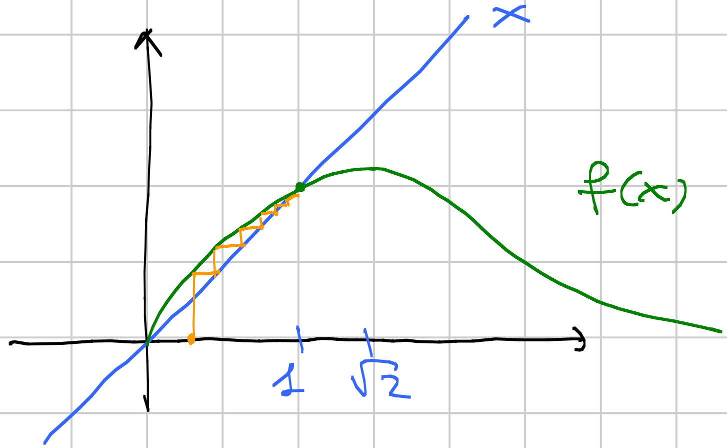
Studio $f(x) \geq x$

$\frac{3x}{x^2+2} \geq x$; $\frac{3x}{x^2+2} - x \geq 0$

$\frac{3x-x^3-2x}{x^2+2} \geq 0 \qquad \frac{x-x^3}{x^2+2} \geq 0 \qquad \frac{x(1-x^2)}{x^2+2} \geq 0$

Per $x>0$ ho che $f(x) \geq 0 \iff 1-x^2 \geq 0 \iff x \leq 1$

Devo capire dove sta il max di $f(x)$



$$f'(x) = \frac{3(x^2+2) - 2x \cdot 3x}{(x^2+2)^2}$$

$$= \frac{3x^2+6-6x^2}{(x^2+2)^2} = \frac{6-3x^2}{(x^2+2)^2}$$

Il max è per $x^2 = 2$ $\quad x = \sqrt{2}$

$\boxed{\text{PIANO}}$

(i) $0 \leq a_n \leq 1 \quad \forall n \in \mathbb{N}$ $\qquad \frac{1}{2004} \leq a_n \leq 1$ Ok.

(ii) $a_{n+1} \geq a_n \quad \forall n \in \mathbb{N}$

(iii) $a_n \to l \in \mathbb{R}$

(iv) $l = 1$

Fase (iv)

$$a_{n+1} = \frac{3 a_n}{a_n^2 + 2}$$

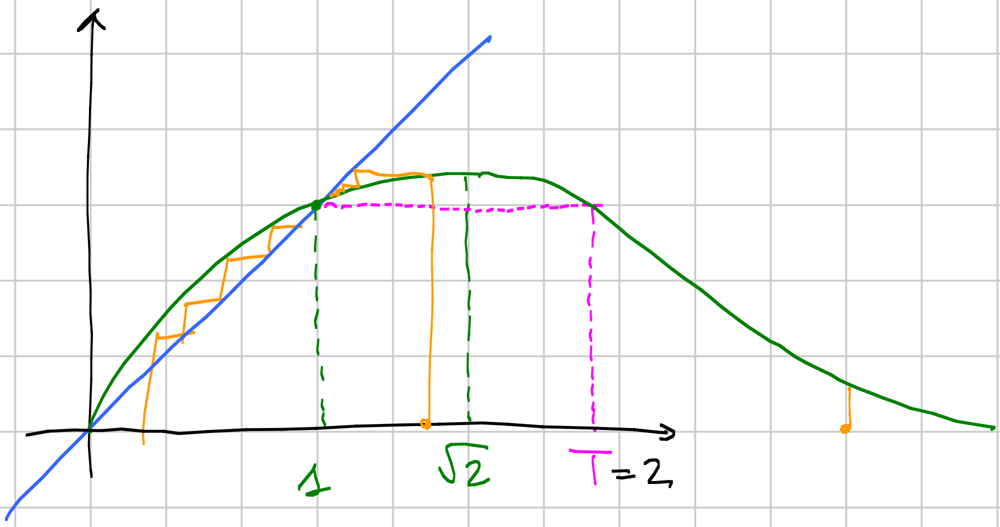
$$\downarrow$$

$$l = \frac{3l}{l^2+2} \Rightarrow l^3 + 2l = 3l$$

$$l^3 - l = 0 \qquad l(l^2-1) = 0 \longrightarrow l = \begin{cases} \boxed{0} \\ 1 \\ \boxed{-1} \leftarrow \text{NO} \end{cases}$$

$\boxed{0} \to$ parte da $\frac{1}{2004}$, poi cresce, quindi è sempre $\geq \frac{1}{2004}$

quindi NO



Tutti gli $\alpha \in (1, \sqrt{2})$ vanno bene

Tutti gli $\alpha \in [0, 1]$ vanno male

Tutti gli $\alpha \geq T$ vanno male

Quelli che vanno bene sono quelli in $(1, T)$

Per calcolare T risolvo $f(x) = 1$

$$\frac{3x}{x^2+2} = 1 \qquad x^2+2 = 3x \qquad x^2-3x+2=0 \qquad (x-1)(x-2)$$

$$x=1 \qquad x=2$$