RICEVIMENTO 8

Titolo nota 06/12/2007

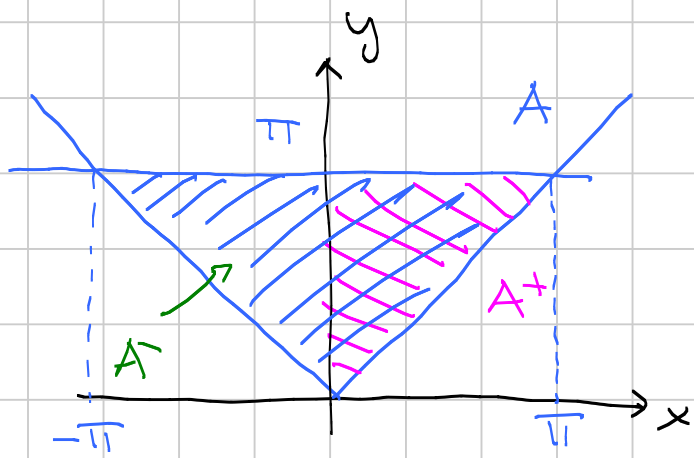
PAG 105

$$\iint_A |x \sin y| \, dx\, dy \qquad A = \{(x,y) \in \mathbb{R}^2 : |x| \le y \le \pi\}$$

$x \sin y$ ↗ ≥ 0 in A^+

$x \sin y$ ↘ ≤ 0 in A^-

sempre ≥ 0 per $y \in [0,\pi]$



$f(x,y)$ è pari risp. alla variabile x $\qquad f(x,y) = f(-x,y)$

$$\iint_A |x \sin y| \, dx\, dy = 2 \iint_{A^+} x \sin y \, dx\, dy = 2\int_0^\pi dx \int_x^\pi dy \; x \sin y$$

$$= 2\int_0^\pi x\, dx \int_x^\pi \sin y \, dy = 2\int_0^\pi x\, dx \left[-\cos y\right]_{y=x}^{y=\pi} = 2\int_0^\pi x(1+\cos x)\, dx$$

Bovinamente ...

$$\iint_A = \iint_{A^+} + \iint_{A^-} = \iint_{A^+} x \sin y \, dx\, dy - \iint_{A^-} x \sin y \, dx\, dy$$

$$= \int_0^\pi dx \int_x^\pi dy \; x \sin y \; - \int_{-\pi}^0 dx \int_{-x}^\pi dy \; x \sin y$$

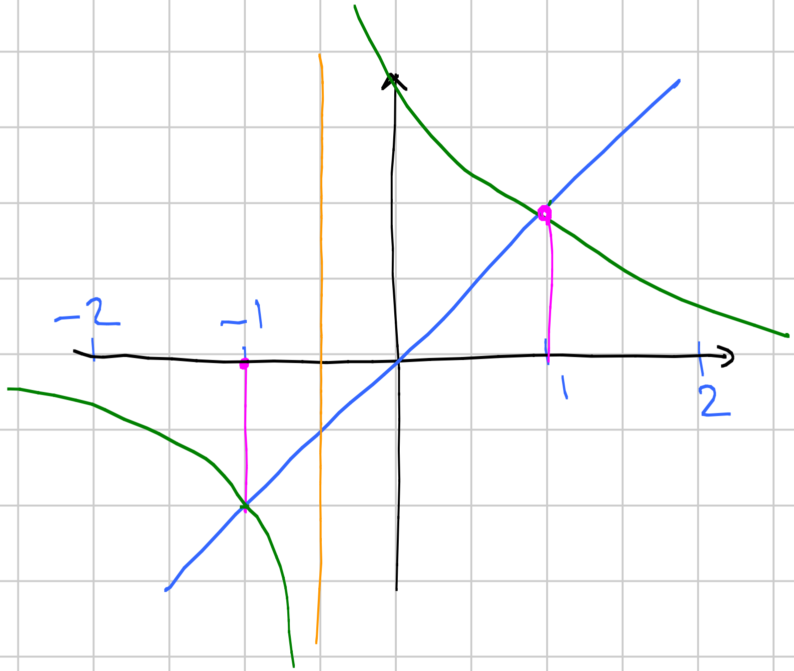
— o — o —

$\boxed{\text{PAG 32}}$ $\quad a_{n+1} = \dfrac{a_n + 2}{2a_n + 1} \qquad a_0 = -1 \qquad a_n = -1 \quad \forall n \in \mathbb{N}.$

$$f(x) = \frac{x+2}{2x+1}$$

Induzione $\quad n=0 \quad$ OK

P.I. ...



$\boxed{\text{PAG 92}}$ $\qquad a_{n+1} = \sqrt[n]{1+a_n} \qquad a_1 = 2007$

$\boxed{\text{PIANO}}$ (i) $1 \le a_n \le 10.000 \quad \forall n \in \mathbb{N}$

(ii) $a_n \to 1$

Dim di (ii) dato per buono (i)

$$\underset{\substack{\downarrow\\ 1}}{\boxed{\sqrt[n]{2}}} \le \underset{\substack{\downarrow\\ 1}}{\boxed{\sqrt[n]{1+a_n}}} \le \underset{\substack{\downarrow\\ 1}}{\boxed{\sqrt[n]{10.000}}}$$

Dim di (i): INDUZIONE $\qquad n=1$ è ovvio

$\boxed{\text{P.I.}}$ Ipotesi $1 \le a_n \le 10.000 \qquad$ Tesi $1 \le a_{n+1} \le 10.000$

Dim. Se $1 \le a_n \le 10.000$, allora $2 \le 1+a_n \le 10.001$

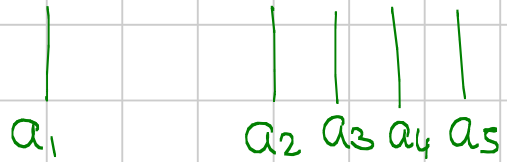
$$1 \le \sqrt[n]{2} \le \sqrt[n]{1+a_n} \le \sqrt[n]{10.001} \le \sqrt{10.001} = 100,\ldots \le 10000$$

$$1 \le \quad a_{n+1} \quad \le 10.000$$

FALSA per $n=1$

Il P.I. vale da $n = 2$ in poi, quindi DEVO controllare a_2 a mano !!!!

Ma $a_2 = 2008$, e va benissimo.



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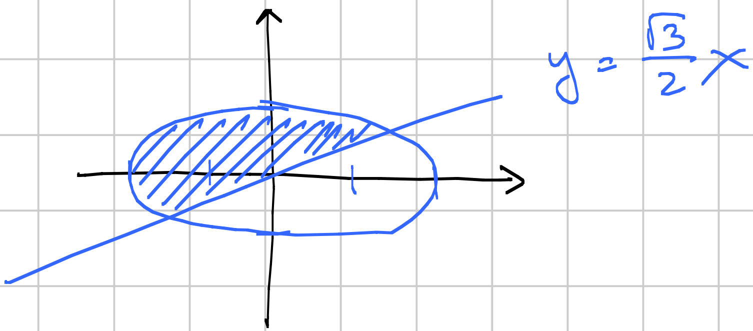
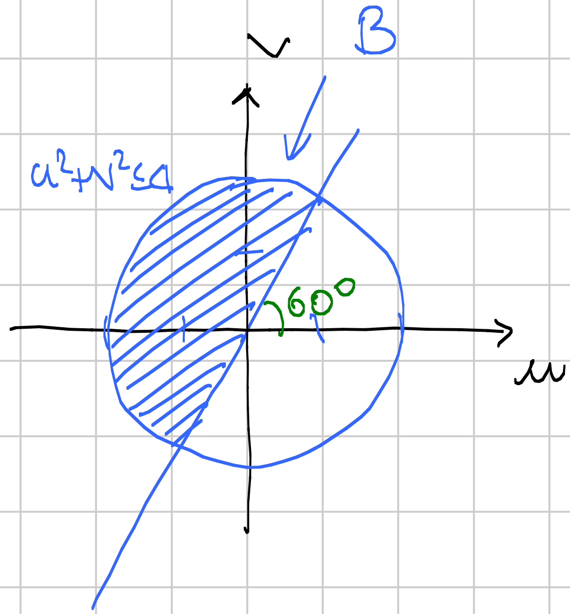
$\boxed{\text{PAG. 106}}$ $\quad \displaystyle\iint_A x\,dx\,dy \qquad A = \{(x,y) \in \mathbb{R}^2 : \underbrace{x^2+4y^2 \le 4}_{\text{ELLISSE}},\ \underbrace{\frac{\sqrt{3}}{2}x \le y}_{\text{SOPRA LA RETTA}}\}$



$$\underbrace{x}_{u}^2 + (\underbrace{2y}_{v})^2 \le 4 \qquad \sqrt{3}\,x \le 2y$$

$$x = u \qquad 2y = v$$

$$u^2 + v^2 \le 4 \qquad \sqrt{3}\,u \le v$$



$$\iint_A x\,dx\,dy = \iint_B \overbrace{u}^{x}\ \overbrace{\frac{1}{2}}^{J}\ du\,dv$$

$$= \frac{1}{2}\int_0^2 d\rho \int_{\frac{\pi}{3}}^{\frac{4\pi}{3}} d\theta\ \overbrace{\rho\cos\theta}^{u}\cdot\overbrace{\rho}^{J}$$

$$= \frac{1}{2}\int_0^2 \rho^2\,d\rho \int_{\frac{\pi}{3}}^{\frac{4\pi}{3}} \cos\theta\,d\theta = \frac{1}{2}\int_0^2 \rho^2\,d\rho\,\Big[\sin\theta\Big]_{\theta=\frac{\pi}{3}}^{\theta=\frac{4\pi}{3}}$$

$$= \frac{1}{2}\int_0^2 \rho^2\,d\rho\left[-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right] = -\frac{\sqrt{3}}{2}\int_0^2 \rho^2\,d\rho = -\sqrt{3}\cdot\frac{1}{\cancel{2}}\cdot\frac{\cancel{8}^{\,4}}{3} = -\frac{4\sqrt{3}}{3}$$

$\boxed{\text{PAG 104}}$

$$\iint_A \frac{x}{1+x^2y^2}\,dx\,dy \qquad A = [0,1]\times[0,1]$$

$$= \int_0^1 dx \int_0^1 dy \frac{x}{1+x^2y^2} = \int_0^1 dx \left[\arctan(xy)\right]_{y=0}^{y=1}$$

$$= \int_0^1 \arctan x \, dx = \text{viene}$$

$$\int \frac{a}{1+a^2y^2}\,dy \rightsquigarrow \int \frac{a}{1+a^2x^2}\,dx$$

— o — o —

$\boxed{\text{PAG. 91}}$ $\quad a_{n+1} = \dfrac{a_n}{n} + \sqrt{n} \qquad a_1 = 3$

$\boxed{\text{PIANO}}$ (i) $a_n \geq 0 \quad \forall n \in \mathbb{N}$ Banale per induzione

(ii) $a_n \to +\infty$ $\quad \underbrace{\boxed{a_{n+1}}}_{+\infty} \geq \underbrace{\boxed{\sqrt{n}}}_{+\infty}$ Se $a_{n+1} \to +\infty$, anche $a_n \to +\infty$

È chiaro che $\sum a_n$ diverge a $+\infty$ perchè
- è a termini ≥ 0
- non verifica la condizione necessaria.

Più interessante : $\sum \frac{1}{a_n}$

$\frac{1}{a_n} \to 0$ quindi ok la cond. nec.

Abbiamo dim. che $a_n \geq \sqrt{n-1}$, quindi

$\frac{1}{a_n} \leq \frac{1}{\sqrt{n-1}}$ $\quad \sum \frac{1}{\sqrt{n-1}}$ diverge $\Rightarrow$ BOH !!

Dico che $a_n \leq n \quad \forall n \in \mathbb{N}$. Se avessi ragione, avrei che $\frac{1}{a_n} \geq \frac{1}{n}$ $\quad \sum \frac{1}{n} = +\infty \Rightarrow \sum a_n = +\infty$

Dimostro $a_n \leq n$ per induz. $\boxed{n=1}$

P.I. Ipotesi $a_n \leq n$
Tesi $a_{n+1} \leq n+1$

$$a_{n+1} = \frac{a_n}{n} + \sqrt{n} \overset{\text{Uso Hp}}{\leq} \frac{n}{n} + \sqrt{n} = 1 + \sqrt{n} \overset{\sqrt{n} \leq n}{\leq} 1 + n$$

FINE

$$\lim_{x\to 0} \frac{e^{x^2-x} - \cos(x-x^2) + x}{x^2} =$$ (Taylor di ordine 2)

$$e^{x^2-x} = 1 + x^2 - x + \frac{1}{2}(x^2-x)^2 + o(x^2) = 1 + x^2 - x + \frac{1}{2}x^2 + o(x^2)$$

$$e^t = 1 + t + \frac{t^2}{2} + o(t^2)$$

$$= 1 - x + \frac{3}{2}x^2 + o(x^2)$$

$$\cos(x-x^2) = 1 - \frac{1}{2}(x-x^2)^2 + o(x^2) = 1 - \frac{1}{2}x^2 + o(x^2)$$

$$\cos t = 1 - \frac{t^2}{2} + o(t^2)$$

$$\frac{\text{Num}}{\text{Den}} = \frac{\cancel{1} - \cancel{x} + \frac{3}{2}x^2 - \cancel{1} + \frac{x^2}{2} + \cancel{x} + o(x^2)}{x^2} \longrightarrow 2$$

$$\lim_{x\to 0} \frac{e^{x+3}-e^3}{\log(1+3x)} = \lim_{x\to 0} \frac{e^3 e^x - e^3}{\log(1+3x)} = e^3 \lim_{x\to 0} \frac{e^x - 1}{\log(1+3x)}$$

$$= e^3 \lim_{x\to 0} \frac{\cancel{1}+x-\cancel{1}+o(x)}{3x+o(x)} = \frac{e^3}{3}$$

— o —

$$\lim_{x\to 0^+} \frac{x\sqrt{x}+\cos x - 1}{\sin x^5 + \sin^5 x + \sqrt{\arctan(2x^3)}}$$

Sviluppo all'ordine $x^{3/2}$

$$\cos x = 1 - \frac{x^2}{2} + \dots = 1 + o(x\sqrt{x})$$

$$\text{Num} = x\sqrt{x} + \cancel{1} - \cancel{1} + o(x\sqrt{x})$$

$$\sin x^5 = o(x\sqrt{x}) \; ; \; \sin^5 x = o(x\sqrt{x})$$

$$\arctan(2x^3) = 2x^3 + o(x^6) \; ; \; \sqrt{\arctan(2x^3)} = \sqrt{2}\, x\sqrt{x} + o(x\sqrt{x})$$

$$\arctan t = t + o(t^2)$$

$$\frac{\text{Num}}{\text{Den}} = \frac{x\sqrt{x} + o(\ )}{\sqrt{2}\cdot x\sqrt{x} + o(\ )} \to \frac{1}{\sqrt{2}}$$

— o —

$$\lim_{x\to 0} \frac{\log^4(1+x)}{x}$$

$\log(1+x) \sim x$

$\log^4(1+x) \sim x^4$

$$\sim \lim_{x\to 0} \frac{x^4}{x} = 0$$

$$\lim_{x\to 0} \frac{\log^4(1+x)}{x - \arctan(x-x^4)}$$

$\log^4(1+x) \sim x^4$

$\arctan t \sim t$

$\arctan(x-x^4) \sim x - x^4$

$$\frac{\text{Num}}{\text{Den}} \sim \frac{x^4}{\cancel{x} - \cancel{x} + x^4} \to 1$$

NO !!!!!!

$$\log(1+x) = x + o(x) \qquad \log^4(1+x) = x^4 + o(x^4)$$

$$\arctan(x-x^4) = (x-x^4) - \frac{1}{3}(x-x^4)^3 + \dots =$$

$$\arctan t = t - \frac{t^3}{3} + \frac{t^5}{5} + \dots \qquad = x - x^4 - \frac{1}{3}x^3 + o(x^4)$$

Den: $x - \arctan(x-x^4) = \cancel{x} - \cancel{x} + \frac{1}{3}x^3 + o(x^3)$

$$\frac{\text{Num}}{\text{Den}} = \frac{x^4 + o(x^4)}{\frac{x^3}{3} + o(x^3)} = \frac{\cancel{x^4}\left(1 + \frac{o(x^4)}{x^4}\right)}{\cancel{x^3}\left(\frac{1}{3} + \frac{o(x^3)}{x^3}\right)} \to 0$$

— o —

$x \to 0$

$$\frac{1-\cos^2 x^3}{1-\cos^3 x^2} = \frac{(1-\cos x^3)\boxed{(1+\cos x^3)}}{(1-\cos x^2)\boxed{(1+\cos x^2+\cos^2 x^2)}} \to 0$$

$\downarrow \frac{2}{3}$

$$\frac{1-\cos x^3}{1-\cos x^2} = \underbrace{\boxed{\frac{1-\cos x^3}{x^6}}}_{\downarrow \frac{1}{2}} \underbrace{\boxed{\frac{x^4}{1-\cos x^2}}}_{\downarrow 2} \cdot \underbrace{\boxed{x^2}}_{\downarrow 0}$$

$$\frac{1-\cos^2 x^3}{1-\cos^3 x^2} \cdot \underbrace{\boxed{\cos(\text{Mostro})}}_{\text{compreso tra } -1 \text{ e } 1} \to 0$$

Criterio del valore assoluto $\quad f(x) \to 0 \iff |f(x)| \to 0$

$$\underbrace{0}_{\downarrow\, 0} \le \underbrace{|f(x) \cdot \cos(\text{Mostro})|}_{\downarrow\, 0} \le \underbrace{|f(x)|}_{\downarrow\, 0}$$

$0 \Rightarrow$ tende a 0 anche senza $|\cdot|$

—o—

$$\sum \underbrace{(-1)^n \frac{1}{n+2} \sin\left(\frac{8}{n}\right)}_{a_n} \qquad |a_n| = \frac{1}{n+2} \underbrace{\sin\left(\frac{8}{n}\right)}_{>0 \text{ per } n \text{ grande}}$$

$$|a_n| \sim \frac{1}{n^2} \Rightarrow \sum |a_n| \text{ conv.} \Rightarrow \sum a_n \text{ conv.}$$

$$\lim_{x \to -\infty} \frac{e^{x+\sin x}-1}{x} = 0$$

$x + \sin x \to -\infty$

$e^{-\infty} = 0$

Num $\to -1$

Den $\to -\infty$

$\Rightarrow$ Frazione $\to 0^+$.

$$\lim_{x \to +\infty} \frac{e^{x+\sin x}-1}{x} = +\infty$$

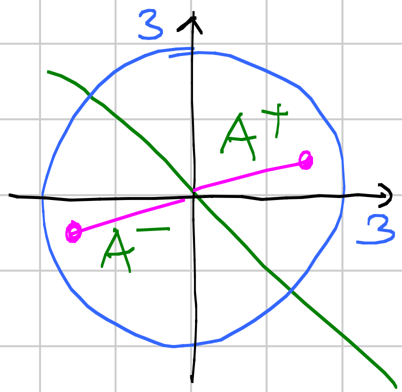
$$\lim_{x \to 0} \frac{e^{x+\sin x}-1}{x}$$

$$\frac{e^{x+\sin x}-1}{x} \sim \boxed{\frac{e^{2x}-1}{2x}}\, 2 \sim 2$$

$\uparrow$ 1

$\boxed{\text{PAG } 106}$

$$\iint_A |x+y| \, dx\,dy$$



$$|x+y| = \begin{cases} x+y & \text{dove } x+y \geq 0, \text{ cioè } y \geq -x \quad A^+ \\ -x-y & \text{"} \quad x+y \leq 0, \text{ "} \quad y \leq -x \quad A^- \end{cases}$$

$$\iint_{A^+} (x+y)\,dx\,dy - \iint_{A^-} (x+y)\,dx\,dy =$$

$$= \int_0^3 d\rho \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} d\theta \; \rho(\cos\theta + \sin\theta) \underset{J}{\rho} - \int_0^3 d\rho \int_{\frac{3\pi}{4}}^{\frac{7\pi}{4}} d\theta \; \rho(\cos\theta + \sin\theta) \underset{J}{\rho}$$

Basta fare 2 volte il 1°

$f(x,y) = |x+y|$ è PARI rispetto all'origine

$$f(-x,-y) = |-x-y| = |x+y| = f(x,y)$$

— o —

$\boxed{\text{PAG } 106}$

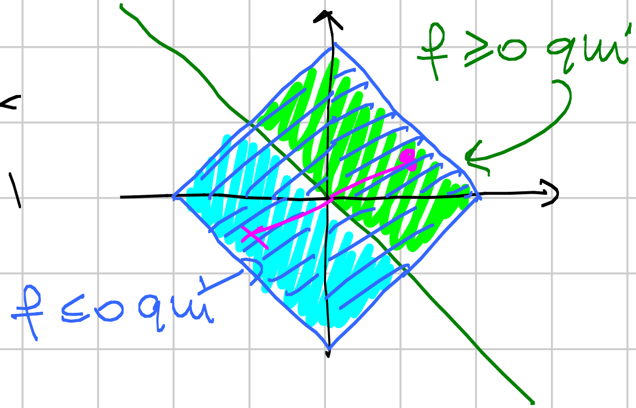
$$\iint_A \arctan(x+y)\,dx\,dy \qquad A = \{(x,y)\in\mathbb{R}^2 : |x|+|y| \le 1\}$$

4 casi: $x \ge 0,\ y \ge 0$ $\quad x+y \le 1$ cioè $y \le 1-x$

II $\to$ $x \le 0,\ y \ge 0$ $\quad -x+y \le 1$ " $\quad y \le x+1$

$x \le 0,\ y \le 0$ ---

$x \ge 0,\ y \le 0$ ---



$\arctan(x+y) = f(x,y)$ è DISPARI RISPETTO ALL'ORIGINE

$$f(-x,-y) = \arctan(-x-y) = -\arctan(x+y) = -f(x,y)$$

$$\iint_A \cos(xy)\,dx\,dy = 4 \cdot \iint_\Delta \cos(xy)\,dx\,dy$$

— o — o —

$\boxed{\text{PAG. 81}}$

$$\sum_{n=1}^{\infty} \frac{\alpha^n}{1+\alpha^{3n}} \qquad \alpha \geq 0$$

Se $\alpha > 1$ $\quad \dfrac{\alpha^n}{1+\alpha^{3n}} \sim \dfrac{\alpha^n}{\alpha^{3n}} = \dfrac{1}{\alpha^{2n}} = \left(\dfrac{1}{\alpha^2}\right)^n$

geometrica con ragione $\frac{1}{\alpha^2} < 1 \Rightarrow$ converge

Se $\alpha = 1$ $\quad \sum \frac{1}{2} \Rightarrow$ Diverge

Se $0 \leq \alpha < 1$ $\quad \dfrac{\alpha^n}{1+\alpha^{3n}} \sim \dfrac{\alpha^n}{1}$ geom. con ragione $\alpha < 1$

$\Rightarrow$ converge