MATEMATICA I

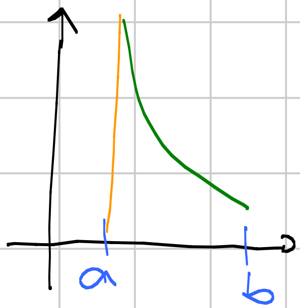
ORA 85

Titolo nota 06/12/2007

Integrali impropri

$$\int_a^{+\infty} f(x)\,dx = \lim_{A\to+\infty} \int_a^A f(x)\,dx$$

$$\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0^+} \int_{a+\varepsilon}^b f(x)\,dx$$



Analogo se il problema è in b.

Esempio $\displaystyle\int_0^{+\infty} x e^{-x}\,dx = \lim_{A\to+\infty} \int_0^A x e^{-x}\,dx =$

$$\int \underset{G}{x}\,\underset{f}{e^{-x}}\,dx = -\underset{G}{x}\,\underset{F}{e^{-x}} + \int \underset{g}{1}\cdot\underset{F}{(+e^{-x})}\,dx = -xe^{-x} - e^{-x}$$

$$\lim_{A\to+\infty}\left[-xe^{-x}-e^{-x}\right]_{x=0}^{x=A} = \lim_{A\to+\infty}\Big(\underset{\substack{\downarrow\\ 0}}{-Ae^{-A}} - \underset{\substack{\downarrow\\ 0}}{e^{-A}} + 1\Big) = 1$$

$\boxed{\text{CRITERI}}$ Integranda $f(x)$ a segno qualunque $\rightarrow$ Assoluta integrab.

$$\int_E |f(x)|\,dx \text{ conv.} \Rightarrow \int_E f(x)\,dx \text{ converge}$$

$f(x)$ segno costante
- $\nearrow$ criterio confronto
- $\searrow$ criterio confronto asint
  - $\nearrow$ casi standard
  - $\searrow$ casi limite

Tabellina di integrali impropri

$$\int_1^{+\infty} \frac{1}{x^\alpha}\,dx \quad \begin{cases} \nearrow \text{conv. se } \alpha > 1 \\ \searrow \text{div. a } +\infty \text{ se } \alpha \le 1 \end{cases}$$

$$\int_2^{+\infty} \frac{1}{x \log^\alpha x}\,dx \quad \begin{cases} \nearrow \text{conv. } \alpha > 1 \\ \searrow \text{div. a } +\infty \text{ se } \alpha < 1 \end{cases}$$

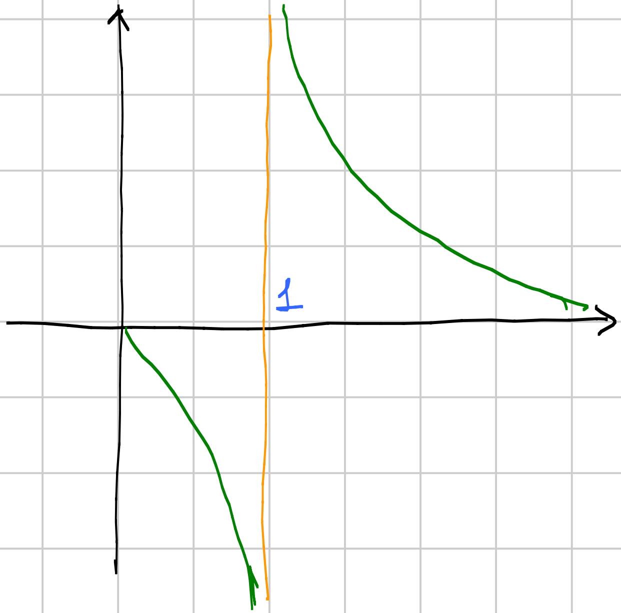
Si usa la definiz.
e si fa la primitiva

$$\int \frac{1}{x \log^\alpha x} = \frac{-1}{\alpha - 1} \frac{1}{\log^{\alpha-1} x}$$

fare la verifica (se $\alpha \neq 1$)

$$\int_0^1 \frac{dx}{x^\alpha} \quad \begin{cases} \nearrow \text{conv. se } \alpha < 1 \\ \searrow \text{div. a } +\infty \text{ se } \alpha \ge 1 \end{cases}$$

Esempio $\displaystyle\int_1^5 \frac{1}{\log x}\,dx =$



Problema in $x=1$,
Pongo $y = x-1$ $\quad dy = dx$

Quando $x=1$, ho che $y=0$
" $x=5$, " " $y=4$

$$= \int_0^4 \underbrace{\boxed{\frac{1}{\log(y+1)}}}_{f(y)} dy$$

$\log(1+y) \sim y$ per $y \to 0^+$

quindi $\displaystyle\frac{1}{\log(y+1)} \sim \frac{1}{y}$

quindi $\displaystyle\int \frac{1}{\log(1+y)}\,dy$ con pb. in $y=0$ si comporta come

$\displaystyle\int \frac{1}{y}\,dy$ con pb. in $y=0$ $\Rightarrow$ diverge

Esempio $\displaystyle\int_{1/2}^{2} \frac{dy}{\log y} = \underbrace{\int_{1/2}^{1} \frac{dy}{\log y}}_{\text{Diverge a } -\infty} + \underbrace{\int_{1}^{2} \frac{dy}{\log y}}_{\text{Diverge a } +\infty} = \text{INDETERMINATO}$

Esempio $\displaystyle\int_{10}^{+\infty} \sin\left(\frac{1}{x^2}\right) dx$ Unico problema a $+\infty$

$\sin\left(\frac{1}{x^2}\right)$ ha segno costante >0 quando $x \to +\infty$

Brutale $\sin\left(\frac{1}{x^2}\right) \sim \frac{1}{x^2}$ per $x \to +\infty$

$\int \sin\left(\frac{1}{x^2}\right) dx$ con pb. a $+\infty$ si comporta come

$\int \frac{1}{x^2} dx$ con pb. a $+\infty$ $\Rightarrow$ converge

Rigoroso: confronto asint. fra $f(x) = \sin\frac{1}{x^2}$ e $g(x) = \frac{1}{x^2}$

$$\lim_{x\to \boxed{+\infty}} \frac{f(x)}{g(x)} = \lim_{x\to+\infty} \frac{\sin\left(\frac{1}{x^2}\right)}{\frac{1}{x^2}} \overset{y=\frac{1}{x^2}}{=} \lim_{y\to 0^+} \frac{\sin y}{y} = 1 \begin{array}{l} \neq 0 \\ \neq +\infty \end{array}$$

PROBLEMA

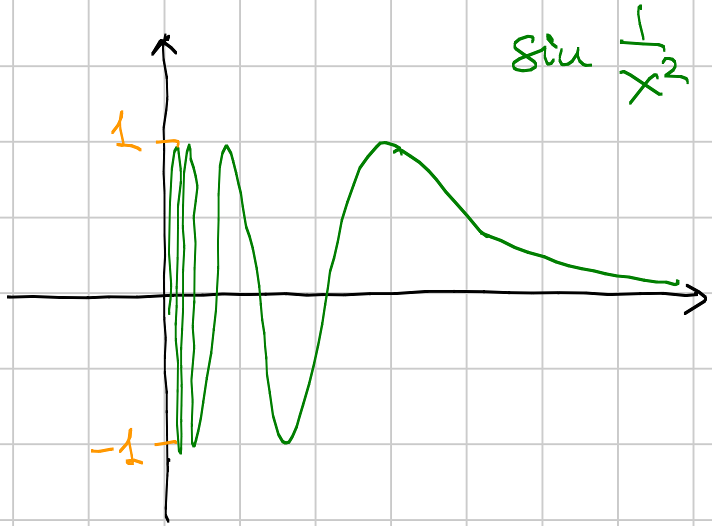
—o—o—

$$\int_0^{+\infty} \sin\left(\frac{1}{x^2}\right) dx$$

Quanti problemi? UNO!!!!! $\to +\infty$

$$\int_0^{10} \sin\left(\frac{1}{x^2}\right) dx$$

è un integrale PROPRIO perché la funzione è LIMITATA



Il comportamento è lo stesso di sopra $\Rightarrow$ converge.

—o—o—

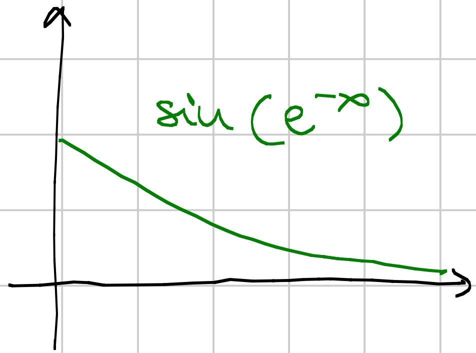
$$\int_0^{+\infty} \sin(e^{-x})\,dx$$

Unico problema: $+\infty$

$f(x) > 0$ quando $x \to +\infty$

e^{-x} assume (per $x \geq 0$) tutti i valori in $(0,1]$, quindi

$\sin(e^{-x})$ è sempre >0 per ogni $x \geq 0$



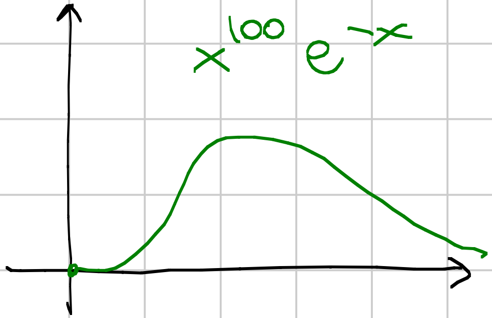
Brutale: $\sin(e^{-x}) \sim e^{-x}$ per $x \to +\infty$

Quindi $\int \sin(e^{-x})\,dx$ con pb. a $+\infty$ si

comporta come $\int e^{-x}\,dx$ con pb. a $+\infty$, quindi converge
(si fa con la definizione)

— o —

$$\int_0^{+\infty} x^{100} e^{-x}\, dx \qquad \text{CONVERGE}$$



RIGOROSO : C.A. con $g(x) = \frac{1}{x^2}$

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = \lim_{x \to +\infty} \frac{x^{100} e^{-x}}{\frac{1}{x^2}} = \lim_{x \to +\infty} x^{102} e^{-x}$$

$$= \lim_{x \to +\infty} \frac{x^{102}}{e^x} = 0$$ CASO LIMITE

Poichè $\lim = 0$, allora $\frac{f(x)}{g(x)} < 1$ per x grandi,

quindi $f(x) < g(x)$ per x grandi $(g(x) > 0)$

quindi $\int f(x)\, dx < \int g(x)\, dx$

CONVERGE

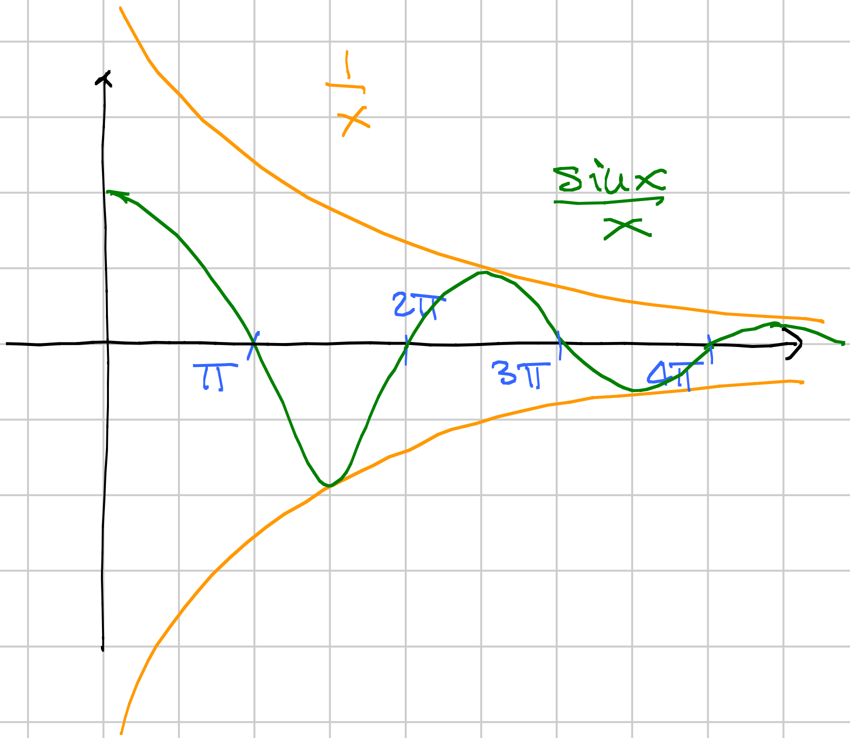
In generale $\displaystyle\int_0^{+\infty} x^\alpha e^{-x}\,dx$ converge $\forall \alpha > 0$.

— o — o —

$\boxed{\text{TRUCCO INTEGRAZIONE PER PARTI}}$

$$\int_1^{+\infty} \frac{\sin x}{x}\,dx$$

Unico problema a $+\infty$.
Integranda ha segno variabile



Unica speranza finora è assol. integ.

$$\int \frac{|\sin x|}{x}\,dx$$

$$\frac{|\sin x|}{x} \leq \frac{1}{x}$$

$$\int_1^{+\infty} \frac{1}{x}\,dx = +\infty \Rightarrow \text{BOH !!}$$

$$\int_1^{+\infty} \frac{\sin x}{x}\,dx = \lim_{A \to +\infty} \int_1^A \frac{\sin x}{x}\,dx$$

$$= \lim_{A \to +\infty} \int_1^A \underset{G}{\frac{1}{x}} \cdot \underset{f}{\sin x}\,dx$$

$$= \lim_{A \to +\infty} \left\{ \left[\frac{\underset{G}{-}\cos x}{\underset{F}{x}} \right]_{x=1}^{x=A} - \int_1^A \underset{g}{\left(-\frac{1}{x^2}\right)} \underset{F}{(-\cos x)}\,dx \right\}$$

$$= \lim_{A \to +\infty} \left\{ -\underset{\downarrow \, 0}{\boxed{\frac{\cos A}{A}}} + \frac{\cos 1}{1} - \int_1^A \frac{\cos x}{x^2}\,dx \right\}$$

$$= \frac{\cos 1}{1} - \underbrace{\int_1^{+\infty} \frac{\cos x}{x^2}\,dx}_{\text{CONVERGE}}$$

$$= \text{CONVERGE}$$

$$\frac{|\cos x|}{x^2} \le \frac{1}{x^2}$$

$\int \frac{1}{x^2}\,dx$ con pb. a $+\infty$ converge

$\Downarrow$ CONFRONTO TRA INTEGRANDE POSITIVE

$\int \frac{|\cos x|}{x^2}\,dx$ con pb. a $+\infty$ converge

$\Downarrow$ ASSOLUTA INTEGRABILITÀ

$\int \frac{\cos x}{x^2}\,dx$ con pb. a $+\infty$ converge

Oss. Anche $\int_0^{+\infty} \frac{\sin x}{x}\,dx$ converge (in $x=0$ il problema non c'è !!!)