MATEMATICA I

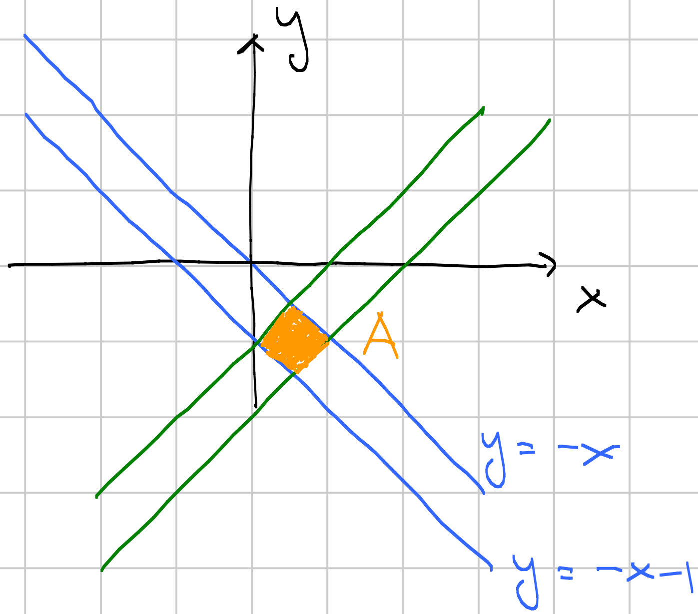
ORA 79

Titolo nota — 30/11/2007

CAMBI DI VARIABILE NEGLI INTEGRALI DOPPI

<u>Esempio</u> $A = \{ (x,y) \in \mathbb{R}^2 : -1 \leq x+y \leq 0,\ 1 \leq x-y \leq 2 \}$

$$\iint_A (x^2-y^2)\,dx\,dy$$



$x+y \leq 0 \qquad y \leq -x$

$x+y \geq -1 \qquad y \geq -x-1$

$x-y \leq 2 \qquad y \geq x-2$

$x-y \geq 1 \qquad y \leq x-1$

$$-1 \leq \underbrace{x+y}_{u} \leq 0 \qquad 1 \leq \underbrace{x-y}_{v} \leq 2$$

$$x^2-y^2 = (x+y)(x-y) = uv$$

$$\iint_A (x^2-y^2)\,dx\,dy = \underbrace{\int_{-1}^{0} du \int_{1}^{2} dv}_{\text{Descrizione insieme A nelle variabili u e v}} \quad \underbrace{uv}_{\text{integranda } x^2-y^2 \text{ scritta nelle variabili u e v}} \quad \boxed{J(u,v)}$$

PAGAMENTO

Come si calcola $J(u,v)$?

1. Ricavare
2. Matrice
3. Determinante

1.
$$\left.\begin{aligned} u &= x+y \\ v &= x-y \end{aligned}\right] \text{ Quello che ho posto}$$

Ricavare x e y in funzione di u e v

Sommo: $u+v = 2x$

Sottraggo: $u-v = 2y$

$$\boxed{\begin{aligned} x &= \frac{u+v}{2} \\ y &= \frac{u-v}{2} \end{aligned}}$$

② MATRICE JACOBIANA $\quad x_u$ = derivata di x rispetto ad u

$$\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

In questo caso la matrice è costante.
In generale gli elementi saranno funzioni di u e v

③ $J(u,v) = |\mathrm{Det}\,(\text{Matrice})|$

$$\mathrm{Det}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

Nel nostro caso $\quad J(u,v) = \left| -\frac{1}{4} - \frac{1}{4} \right| = \frac{1}{2}$

Conclusione dell' esercizio : $\displaystyle\iint_A (x^2-y^2)\,dx\,dy = \int_{-1}^{0} du \int_{1}^{2} uv\,\frac{1}{2}\,dv$

$$= \frac{1}{2}\int_{-1}^{0} u\,du \cdot \int_{1}^{2} v\,dv = \ldots$$

Esempio di calcolo di J — Coordinate polari

① Ricavare $\quad x = \rho\cos\theta \qquad y = \rho\sin\theta$

② Matrice jacobiana $\begin{pmatrix} x_\rho & x_\theta \\ y_\rho & y_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{pmatrix}$

③ Determinante

$$J(\rho,\theta) = |\mathrm{Det}(\text{Matrice})| = |\rho\cos^2\theta + \rho\sin^2\theta| = \rho$$

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Cambi di variabile particolarmente usati

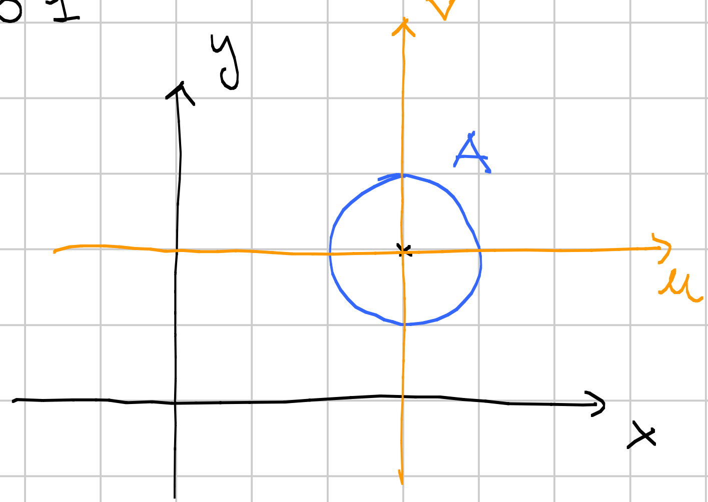
$\boxed{\text{TRASLAZIONI}}$ $\quad u = x - x_0 \qquad v = y - y_0 \qquad x_0, y_0$ numeri fissati

① Ricavare: $x = u + x_0$, $y = v + y_0$

② Matrice: $\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ③ $J(u,v) = 1$

Esempio

$$\iint_A x\,dx\,dy \qquad A = \text{cerchio con centro in } (3,2) \text{ e raggio } 1$$



Equazione di A:

$$\underbrace{(\overbrace{x-3}^{u})^2 + (\overbrace{y-2}^{v})^2}_{\text{distanza di } (x,y) \text{ da } (3,2)} \le \underbrace{1}_{\text{raggio}^2}$$

$u^2+v^2 \le 1$

$$\iint_A x\,dx\,dy = \iint_{\underbrace{u^2+v^2\le 1}_{\text{insieme nelle coordinate } (u,v)}} \underbrace{(u+3)}_{x \text{ scritta in termini di } u \text{ e } v} \cdot \overbrace{1}^{J(u,v)}$$

$$= \iint_{u^2+v^2\le 1} (u+3)\,du\,dv = \overbrace{\iint_{u^2+v^2\le 1} u\,du\,dv}^{=0} + \iint_{u^2+v^2\le 1} 3\,du\,dv = 3\,\text{Area} = 3\pi$$

DILATAZIONI DEGLI ASSI

$$u = ax$$
$$v = by$$

① Ricavare $\quad x = \frac{u}{a} \qquad y = \frac{v}{b}$ $\qquad a>0,\ b>0$

② Matrice

$$\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{pmatrix}$$

③ Determinante $\quad J(u,v) = |\mathrm{Det}| = \frac{1}{ab}$

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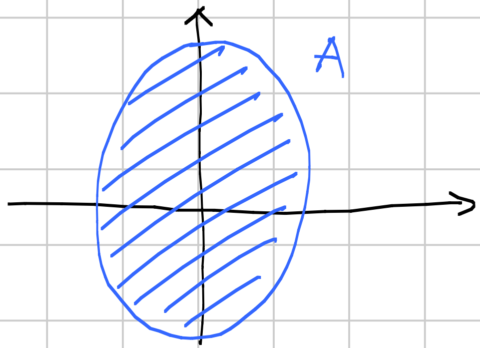
Esempio $\quad A = \{(x,y) \in \mathbb{R}^2 ;\ 3x^2 + 2y^2 \leq 5\}$

$$\mathrm{Area}(A) = \iint_A 1\, dx\, dy$$

$$\underbrace{(\sqrt{3}x)}_{u}{}^2 + \underbrace{(\sqrt{2}y)}_{v}{}^2 \leq 5$$

$$u^2 + v^2 \leq 5$$



$$\iint_A 1\,dx\,dy = \iint_{\underbrace{u^2+v^2\le 5}_{\text{INSIEME A nelle coordinate u e v}}} \underbrace{1}_{\text{funzione resta uguale}} \cdot \underbrace{\frac{1}{\sqrt{2}\sqrt{3}}}_{\text{J calcolato con } a=\sqrt{2},\ b=\sqrt{3}} \,du\,dv$$

$$= \frac{1}{\sqrt{6}} \iint_{u^2+v^2\le 5} 1\,du\,dv$$

$$= \frac{1}{\sqrt{6}} \text{Area}\left(\text{cerchio } u^2+v^2\le 5\right)$$

$$= \frac{1}{\sqrt{6}} \pi\, 5 = \frac{5\pi}{\sqrt{6}}$$

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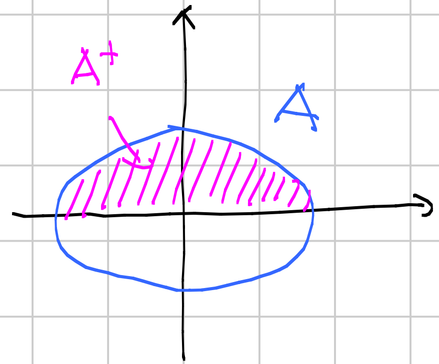
$$\underbrace{\left(\frac{x}{A}\right)}_{=u}^2 + \underbrace{\left(\frac{y}{B}\right)}_{=v}^2 \le 1$$

In generale l'area dell'ellisse in forma canonica $\frac{x^2}{A^2}+\frac{y^2}{B^2}\le 1$

è data dalla formula $\pi A B$ (se $A=B$ è l'area del cerchio)

Altro esempio

$$A = \{(x,y) \in \mathbb{R}^2 : \quad x^2 + 3y^2 \le 5\}$$



$$\iint_A y \, dx\, dy = 0$$

$$\iint_A |y| \, dx\, dy = 2 \iint_{A^+} y \, dx\, dy =$$

$x^2 + 3y^2 \le 5$

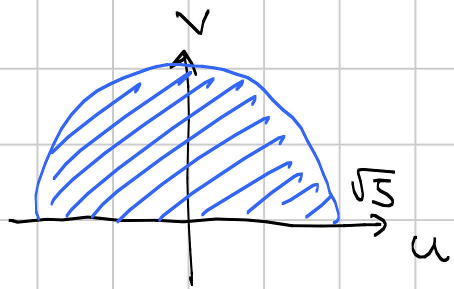
$x^2 + (\sqrt{3}y)^2 \le 5$

$u = x \qquad v = \sqrt{3}\, y$

$a = 1 \qquad b = \sqrt{3}$

$$= 2 \underbrace{\iint_{\substack{u^2+v^2 \le 5 \\ v \ge 0}}}_{A^+ \text{ nelle nuove coord}} \underbrace{\frac{v}{\sqrt{3}}}_{y} \; \underbrace{\frac{1}{\sqrt{3}}}_{J(u,v) = \frac{1}{ab}} \, du\, dv$$

$$= \frac{2}{3} \iint_{\substack{u^2+v^2 \le 5 \\ v \ge 0}} v \, du\, dv$$



$u = \rho \cos\theta$

$v = \rho \sin\theta$

$$= \frac{2}{3} \underbrace{\int_0^{\sqrt{5}} d\rho \int_0^{\pi} d\theta}_{\text{semicerchio}} \underbrace{\rho \sin\theta}_{V} \underbrace{\rho}_{J} =$$

$$= \frac{2}{3} \int_0^{\sqrt{5}} \rho^2 d\rho \underbrace{\int_0^{\pi} \sin\theta \, d\theta}_{2} = \frac{4}{3} \int_0^{\sqrt{5}} \rho^2 d\rho$$

$$= \frac{4}{3} \left[\frac{\rho^3}{3} \right]_{\rho=0}^{\rho=\sqrt{5}} =$$

$$= \frac{20\sqrt{5}}{9}$$