MATEMATICA I

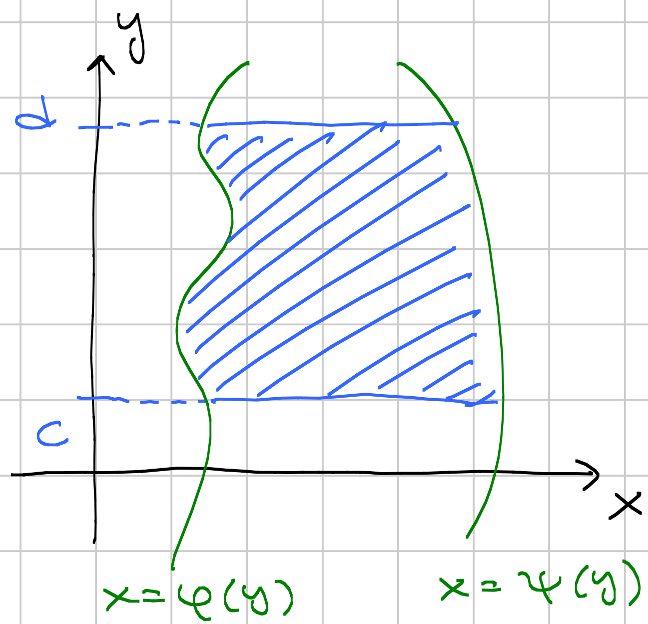
$\boxed{\text{ORA } 75}$

Titolo nota 28/11/2007

INSIEMI NORMALI RISPETTO ASSE y

<u>Def.</u> Un insieme si dice normale rispetto all'asse y se si scrive nella forma

$$A = \{(x,y) \in \mathbb{R}^2 : y \in [c,d], \ \varphi(y) \le x \le \psi(y)\}$$



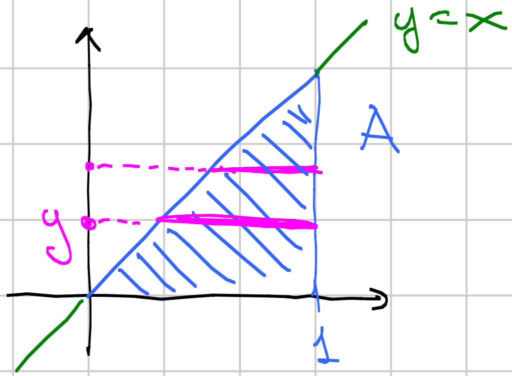
Formula di riduzione:

$$\iint_A f(x,y)\,dx\,dy =$$

$$= \int_c^d dy \int_{\varphi(y)}^{\psi(y)} f(x,y)\,dx$$

Esempio 1

$$A = \{ (x,y) \in \mathbb{R}^2 ; \ y \in [0,1], \quad \underset{\varphi(y)}{y} \le x \le \underset{\psi(y)}{1} \}$$



$$\iint_A x y^2 \, dx\, dy = \int_0^1 dy \int_{\boxed{y}}^{\boxed{1}} x \boxed{y^2} \, dx = \int_0^1 y^2 dy \int_y^1 x \, dx$$

$\psi(y)$; $\varphi(y)$

$$= \int_0^1 y^2 dy \left[\frac{x^2}{2} \right]_{x=y}^{x=1} = \frac{1}{2} \int_0^1 y^2 \{ 1 - y^2 \} \, dy$$

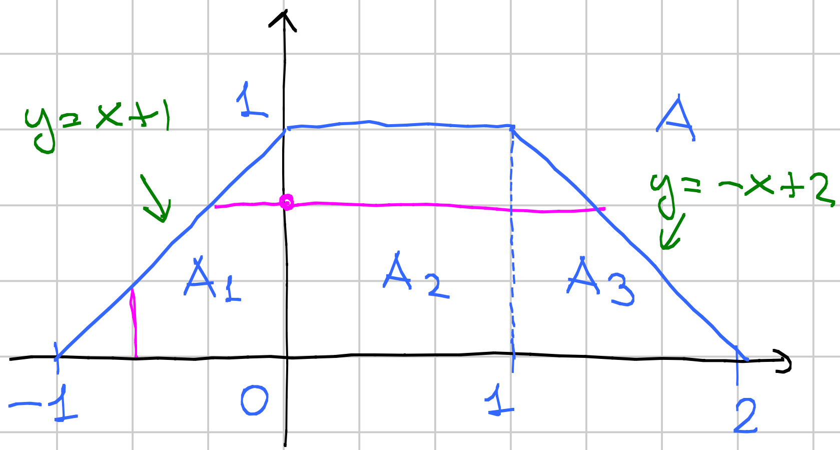
primitiva in x ; $x=1$; $x=y$

$$= \frac{1}{2} \int_0^1 (y^2 - y^4) \, dy = \frac{1}{2} \left[\frac{y^3}{3} - \frac{y^5}{5} \right]_{y=0}^{y=1} = \frac{1}{2} \left(\frac{1}{3} - \frac{1}{5} \right)$$

$$= \frac{1}{15}$$

Esempio 2

$$\iint_A y\,dx\,dy =$$



$$= \iint_{A_1} y\,dx\,dy + \iint_{A_2} y\,dx\,dy + \iint_{A_3} y\,dx\,dy$$

A_1, A_2, A_3 normali rispetto asse x

$$A_1 = \{ (x,y) \in \mathbb{R}^2 : x \in [-1,0], \quad 0 \le y \le x+1 \}$$

$$A_2 = \{ (x,y) \in \mathbb{R}^2 ; x \in [0,1], \ y \in [0,1] \}$$

$$A_3 = \{ (x,y) \in \mathbb{R}^2 ; x \in [1,2], \quad 0 \le y \le -x+2 \}$$

$$= \int_{-1}^{0} dx \int_0^{x+1} y\,dy + \int_0^1 dx \int_0^1 y\,dy + \int_1^2 dx \int_0^{-x+2} y\,dy \quad = \text{SI FA !!}$$

Alternativa: impostazione come insieme norm. risp. asse y

$$A = \{ (x,y) \in \mathbb{R}^2 : y \in [0,1], \underbrace{y-1}_{} \le x \le \underbrace{2-y}_{} \}$$

ho esplicitato $y = x+1$ risp. ad x

ho esplicitato $y = -x+2$ risp. ad x

$$\iint_A y \, dx\,dy = \int_0^1 dy \int_{y-1}^{2-y} \boxed{y} \, dx = \int_0^1 y \, dy \underbrace{\int_{y-1}^{2-y} dx}_{}$$

lungh. intervallo $= (2-y) - (y-1)$

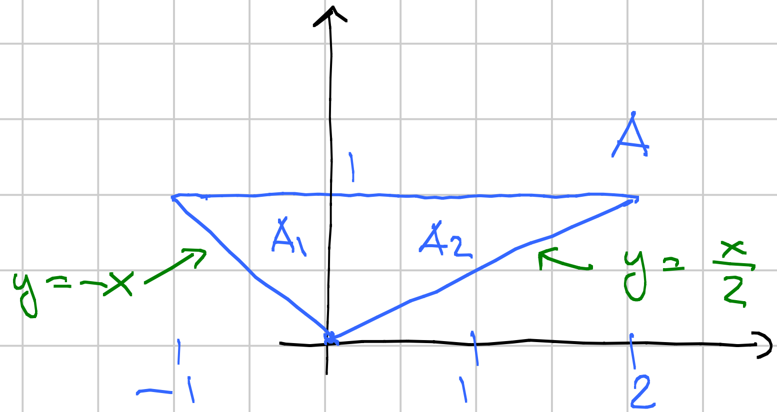
$$= \int_0^1 y(3-2y) \, dy =$$

$$= \int_0^1 (3y - 2y^2) \, dy = \left[\frac{3y^2}{2} - \frac{2y^3}{3} \right]_{y=0}^{y=1} = \frac{3}{2} - \frac{2}{3} = \frac{5}{6}$$

controllare che venga = con l'altro sistema.

Esempio 3

$$\iint_A xy \, dx\, dy =$$



$$= \iint_{A_1} xy \, dx\, dy + \iint_{A_2} xy \, dx\, dy$$

$$= \int_{-1}^{0} dx \int_{-x}^{1} xy \, dy + \int_{0}^{2} dx \int_{x/2}^{1} xy \, dy = \dots \quad \text{calcolare}$$

$A_1 = \{(x,y) \in \mathbb{R}^2 : x \in [-1,0], \quad -x \le y \le 1 \}$

$A_2 = \{(x,y) \in \mathbb{R}^2 : x \in [0,2], \quad \frac{x}{2} \le y \le 1 \}$

A come normale rispetto all'asse y

$A = \{(x,y) \in \mathbb{R}^2 : y \in [0,1], \quad -y \le x \le 2y \}$

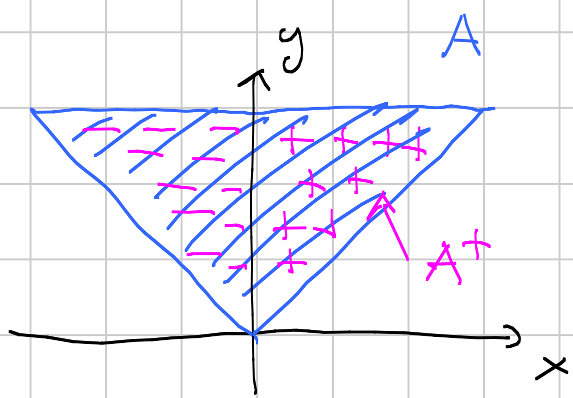
$$\iint_A xy\,dx\,dy = \int_0^1 dy \int_{-y}^{2y} xy\,dx = \int_0^1 y\,dy \int_{-y}^{2y} x\,dx$$

$$= \int_0^1 y\,dy \left[\frac{x^2}{2}\right]_{x=-y}^{x=2y} = \int_0^1 y\left\{ \underset{x=2y}{2y^2} - \underset{x=-y}{\frac{y^2}{2}} \right\} dy$$

$$= \frac{3}{2}\int_0^1 y^3\,dy = \frac{3}{2}\cdot\frac{1}{4} = \frac{3}{8}.$$

— o — o —

$\boxed{\text{Esempio 4}}$ $\displaystyle\iint_A xy\,dx\,dy = 0$



SOLITA RAGIONE DI SIMMETRIA!!!

$f(x,y)$ è DISPARI RISPETTO a x $\quad f(-x,y) = -f(x,y)$

Insieme simmetrico risp. asse y.

$\boxed{\text{Esempio 5}}$ A come in esempio 4

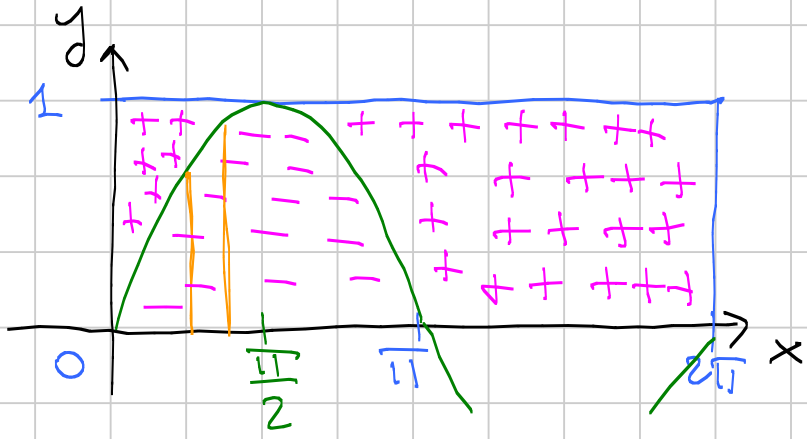
$$\iint_A |xy|\,dx\,dy = 2\iint_{A^+} xy\,dx\,dy = 2\int_0^1 dx \int_x^1 dy\; xy = \text{si finisce}$$

— o — o —

$\boxed{\text{Esempio 6}}$ $A = [0, 2\pi] \times [0, 1]$

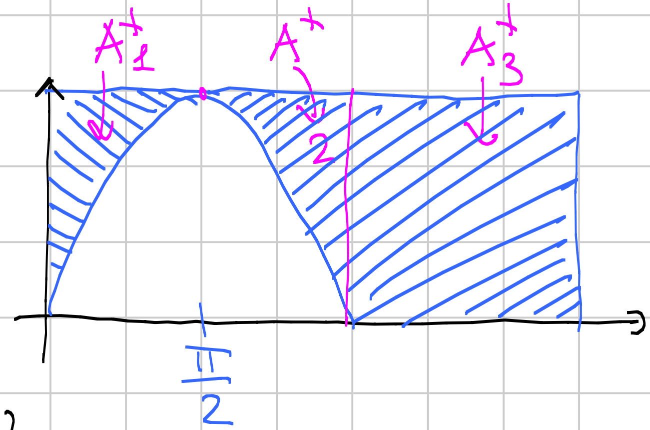
$$\iint_A |y - \sin x|\,dx\,dy = \iint_{A^+} (y - \sin x)\,dx\,dy - \iint_{A^-} (y-\sin x)\,dx\,dy$$

$$|y - \sin x| = \begin{cases} y - \sin x & \text{dove } y - \sin x \geq 0, \text{ cioè } y \geq \sin x \quad A^+ \\ -(y - \sin x) & \text{"} \qquad \leq 0 \qquad \text{"} \quad y \leq \sin x \quad A^- \end{cases}$$



$$A^- = \{ (x,y) \in \mathbb{R}^2 : x \in [0,\pi], \quad 0 \le y \le \sin x \}$$

$$\iint_{A^-} (y - \sin x)\, dx\, dy = \int_0^\pi dx \int_0^{\sin x} (y - \sin x)\, dy = \dots$$



$$A_1^+ = \{ (x,y) \in \mathbb{R}^2 : x \in [0, \tfrac{\pi}{2}], \ \sin x \le y \le 1 \}$$

$$A_2^+ = \{ (x,y) \in \mathbb{R}^2 : x \in [\tfrac{\pi}{2}, \pi], \ \sin x \le y \le 1 \}$$

$$A_3^+ = \{ (x,y) \in \mathbb{R}^2 : x \in [\pi, 2\pi], \ 0 \le y \le 1 \}$$

A_1^+ e A_2^+ si possono mettere insieme $\{ (x,y) \in \mathbb{R}^2 : x \in [0,\pi], \ \sin x \le y \le 1 \}$

Alternativa

$$A^+ = \left([0, 2\pi] \times [0,1] \right) \setminus A^-$$

$$\iint_{A^+} f(x,y)\, dx\, dy = \iint_{\text{rett.}} \quad - \iint_{A^-} f(x,y)\, dx\, dy$$

In conclusione

$$\iint_A |y - \sin x| \, dx\,dy = \iint_{\text{rett.}} (y - \sin x)\,dx\,dy - 2\iint_{A^-} (y - \sin x)\,dx\,dy$$

in soli 2 pezzi

Esercizio: controllare che venga lo stesso !!!