MATEMATICA I

$\boxed{\text{ORA 34}}$

Titolo nota 24/10/2007

$\boxed{\text{Esercizio 1}}$

$$\sum_{n=0}^{\infty} \frac{32 + \cos n! + \arctan n^2}{n^3 + 3} \qquad a_n \geq 0$$

BRUTALE : $a_n \sim \frac{\text{costante}}{n^3} \Rightarrow$ CONVERGE

Rigoroso : confronto (non asintotico)

$$a_n \leq \underbrace{\frac{85}{n^3}}_{b_n} \quad \begin{matrix} \leftarrow \text{Num } + \text{grande} \\ \leftarrow \text{Den. } + \text{piccolo} \end{matrix}$$

$$\sum b_n = \sum \frac{85}{n^3} = 85 \sum \frac{1}{n^3}$$ converge perché armonica gen. con $\alpha = 3 > 1$

$\sum b_n$ converge $\Rightarrow \sum a_n$ converge

Esercizio 2

$$\sum_{n=5}^{+\infty} \frac{12+\cos n^2}{n^2-6} \qquad a_n \geq 0$$

$$\underbrace{\frac{12+\cos n^2}{n^2-6}}_{a_n} \leq \underbrace{\frac{15}{n^2-6}}_{b_n}$$

$\sum b_n$ non è immediatamente un' armonica gen.

Studio a parte $\sum_{n=5}^{\infty} \underbrace{\frac{1}{n^2-6}}_{b_n}$ Per questa faccio il confronto asintotico con $\sum \underbrace{\frac{1}{n^2}}_{c_n}$

$$\lim_{n\to+\infty} \frac{b_n}{c_n} = \lim_{n\to+\infty} \frac{\frac{1}{n^2-6}}{\frac{1}{n^2}} = \lim_{n\to+\infty} \frac{n^2}{n^2-6} = 1 \quad \begin{matrix}\neq 0\\ \neq +\infty\end{matrix}$$

$\sum b_n$ si comporta come $\sum c_n$

$\sum c_n$ converge $\left(\begin{matrix}\text{armonica}\\ \text{gen. con}\\ \alpha=2\end{matrix}\right) \underset{\text{C.A.}}{\Longrightarrow} \sum b_n$ conv. $\overset{\text{confronto}}{\Longrightarrow} \sum a_n$ conv.

Esercizio 3

$$\sum_{n=0}^{\infty} \left(\sqrt{n+2} - \sqrt{n+1} \right) \qquad a_n > 0$$

$$a_n = \sqrt{n+2} - \sqrt{n+1} \quad \frac{\sqrt{} + \sqrt{}}{\sqrt{} + \sqrt{}} = \frac{\cancel{n} + 2 - \cancel{n} - 1}{\sqrt{n+2} + \sqrt{n+1}} = \frac{1}{\sqrt{n+2} + \sqrt{n+1}}$$

$a_n \to 0 \Rightarrow$ cond. nec. ok $\Rightarrow$ può convergere

BRUTALE: $a_n \sim \dfrac{1}{2\sqrt{n}}$ armonica generalizzata con $\alpha = \dfrac{1}{2} \Rightarrow$ DIVERGE

RIGOROSO: confronto asintotico con $b_n = \dfrac{1}{\sqrt{n}}$

$$\frac{a_n}{b_n} = \frac{\sqrt{n}}{\sqrt{n+2} + \sqrt{n+1}} \to \frac{1}{2}$$ (Fare x esercizio)

$\dfrac{1}{2}$ $\neq 0$, $\neq +\infty$, quindi $\sum a_n$ si comporta come $\sum \dfrac{1}{\sqrt{n}}$, cioè DIVERGE

Esempio 4 $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ $a_n > 0$

$\frac{1}{n} \in (0,1)$ e in questa zona il sin è > 0

BRUTALE: $a_n = \sin\left(\frac{1}{n}\right) \sim \frac{1}{n} \Rightarrow$ DIVERGE

RIGOROSO: confronto asintotico con $b_n = \frac{1}{n}$

$$\frac{a_n}{b_n} = \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} \underset{\substack{\uparrow \\ x = \frac{1}{n}}}{=} 1 \quad \begin{matrix} \neq 0 \\ \neq +\infty \end{matrix} \Rightarrow \text{stesso comportamento.}$$

Esempio 5 $\displaystyle\sum_{n=1}^{\infty} \left(\sinh\frac{1}{n} - \sin\frac{1}{n}\right)$ $a_n > 0$

(andrebbe dimostrato e un giorno lo faremo)

BRUTALE 1: $\sinh\frac{1}{n} \sim \frac{1}{n}$

$\sin\frac{1}{n} \sim \frac{1}{n}$ $\quad a_n \sim \cancel{\frac{1}{n}} - \cancel{\frac{1}{n}}$ TROPPO BRUTALE

BRUTALE 2: $\sinh\frac{1}{n} \sim \frac{1}{n} + \frac{1}{6n^3}$ $\qquad \sinh x = x + \frac{x^3}{6} + o(x^3)$

$\sin\frac{1}{n} \sim \frac{1}{n} - \frac{1}{6n^3}$ $\qquad \sin x = x - \frac{x^3}{6} + o(x^3)$

$a_n \sim \cancel{\frac{1}{n}} + \frac{1}{6n^3} - \cancel{\frac{1}{n}} + \frac{1}{6n^3} = \frac{1}{3n^3}$ $\Rightarrow$ armonica gen. con $\alpha = 3$
$\Rightarrow$ CONVERGE

RIGOROSO: C.A. con $b_n = \frac{1}{n^3}$

$$\lim_{n\to+\infty} \frac{a_n}{b_n} = \lim_{n\to+\infty} \frac{\sinh\frac{1}{n} - \sin\frac{1}{n}}{\frac{1}{n^3}} =$$

$$= \lim_{x\to 0} \frac{\sinh x - \sin x}{x^3} = \lim_{x\to 0} \frac{\cancel{x} + \frac{x^3}{6} + o(x^3) - \cancel{x} + \frac{x^3}{6}}{x^3} = \frac{1}{3}$$

$\frac{1}{3} \begin{matrix} \neq 0 \\ \neq +\infty \end{matrix}$ $\Rightarrow$ stesso comportamento

Esempio 6 $\quad \sum_{n=1}^{\infty} \frac{\log n}{n}$ $\qquad a_n > 0$ per $n \geq 2$

$$\underbrace{\frac{\log n}{n}}_{a_n} \geq \underbrace{\frac{1}{n}}_{b_n}$$ definitivamente (non appena $\log n \geq 1$)

$\sum b_n$ DIVERGE $(\alpha = 1) \Rightarrow \sum a_n$ diverge

Esempio 7 $\quad \sum_{n=2}^{\infty} \frac{1}{n^2 \log n}$ $\qquad a_n > 0$

$$\underbrace{\frac{1}{n^2 \log n}}_{a_n} \leq \underbrace{\frac{1}{n^2}}_{b_n}$$ (non appena $\log n \geq 1$)

$\sum b_n$ CONVERGE $(\alpha = 2) \Rightarrow \sum a_n$ CONVERGE

Esempio 8

$$\sum_{n=1}^{\infty} \frac{\log n}{n^2} \qquad a_n > 0 \quad \text{per} \quad n \geq 2$$

1° $\dfrac{\log n}{n^2} \geq \dfrac{1}{n^2}$ $\qquad \sum \frac{1}{n^2}$ converge $\Rightarrow$ BOH

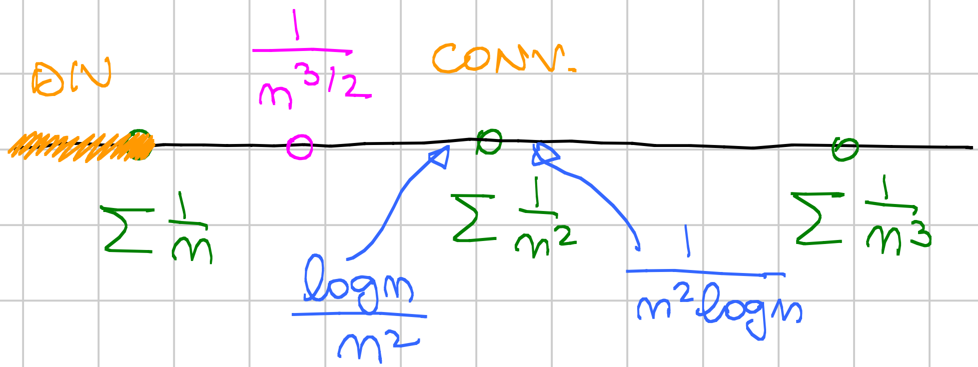
2° C.A. con $b_n = \dfrac{1}{n}$. $\qquad \dfrac{a_n}{b_n} = \dfrac{\log n}{n^2} \cdot n = \dfrac{\log n}{n} \to 0$

Caso limite: $\dfrac{a_n}{b_n} \to 0$, quindi defin. $\dfrac{a_n}{b_n} \leq 1$, cioè

$a_n \leq b_n$

$\sum b_n = \sum \dfrac{1}{n} = +\infty$ (armonica con $\alpha = 1$) $\Rightarrow$ BOH

3°



Confronto asintotico con $b_n = \frac{1}{n^{3/2}} = \frac{1}{n\sqrt{n}}$

$$\frac{a_n}{b_n} = \frac{\frac{\log n}{n^2}}{\frac{1}{n\sqrt{n}}} = \frac{\log n}{n^2} \cdot n\sqrt{n} = \frac{\log n}{\sqrt{n}} \longrightarrow 0$$

caso limite

$\frac{a_n}{b_n} \to 0 \implies \frac{a_n}{b_n} \le 1$ defin. $\implies a_n \le b_n$ defin.

$\sum b_n = \sum \frac{1}{n^{3/2}}$ converge $\left(\alpha = \frac{3}{2} > 1\right) \implies \sum a_n$ converge

—o—

In generale

$$\sum_{n=1}^{\infty} \frac{\log^{30} n}{n^{\alpha}} \quad \text{converge} \iff \alpha > 1$$