MATEMATICA I

ORA 27

Titolo nota 18/10/2007

Giustificazione sviluppi di Taylor

$$P_n(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

<u>Esempio 1</u> $f(x) = e^x$, $f'(x) = e^x$, $f''(x) = e^x$, ..., $f^{(k)}(x) = e^x$

Quindi $f^{(k)}(0) = e^0 = 1$ e pertanto

$$P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

<u>Esempio 2</u> $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$

$f^{IV}(x) = \sin x$, $f^{V}(x) = \cos x$ e così via

$f^{(2007)}(x) = -\cos x$ (periodica di periodo 4)

$f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -1$

$0, 1, 0, -1, 0, 1, 0, -1, \ldots$

$$P_n(x) = 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 - \frac{1}{3!}x^3 + \frac{0}{4!}x^4 + \ldots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

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Esempio 3 $f(x) = \cos x$, $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$
f^{IV} " f^{V} = "

$f(0) = 1$, $f'(0) = 0$, $f''(0) = -1$, $f'''(0) = 0, \ldots$ $1, 0, -1, 0$

$$1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

Esempio 4 $\quad f(x) = \log(1+x) \qquad f'(x) = \dfrac{1}{1+x}$

$$f''(x) = -\frac{1}{(1+x)^2} \qquad f'''(x) = \frac{2}{(1+x)^3}$$

$$f^{IV}(x) = -\frac{3\cdot 2}{(1+x)^4} \qquad f^{V}(x) = \frac{4\cdot 3\cdot 2}{(1+x)^5}$$

In conclusione: $\boxed{f^{(k)}(x) = (-1)^{k+1}\dfrac{(k-1)!}{(1+x)^k}}$ Per esercizio dimostrarla x induzione

$$f^{(k)}(0) = (-1)^{k+1}(k-1)!$$

Il termine di grado k nel polinomio di Taylor è

$$\frac{f^{(k)}(0)}{k!}x^k = (-1)^{k+1}\frac{(k-1)!}{k!}x^k = (-1)^{k+1}\frac{x^k}{k}$$

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots$$

Esempio 1 Calcolare il polinomio $P_5(x)$ per $f(x) = \sin x$

$$P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} = x - \frac{x^3}{6} + \frac{x^5}{120}$$

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)$$

$$f(x) = P_5(x) + o(x^5)$$

Esempio 2 Calcolare $P_5(x)$ per $f(x) = \cos x$

$$P_5(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} = 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5) \qquad f(x) = P_5(x) + o(x^5)$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4) \qquad f(x) = P_4(x) + o(x^4)$$

Esempio 3

$$\sin x = x - \frac{x^3}{6} + o(x^3) \quad \text{VERA} \quad \text{Taylor con } n=3$$

$$\sin x = x - \frac{x^3}{6} + o(x^4) \quad \text{VERA} \quad \text{Taylor con } n=4$$

$$\sin x = x \boxed{- \frac{x^3}{6}} + o(x^2) \quad \text{VERA} \quad \sin x = x + o(x^2)$$

$$\sin x = x - \frac{x^3}{6} + o(x) \quad \text{VERA} \quad \sin x = x + o(x)$$

Esempio 4 $\boxed{x \to 0}$

$$\frac{\sin x - x + x^4}{x^3 + x^4} \overset{\substack{\text{Taylor con} \\ n=3}}{=} \frac{\overbrace{\cancel{x} - \frac{x^3}{6}}^{\sin x} - \cancel{x} + x^4 + o(x^3)}{x^3 + o(x^3)}$$

$$\frac{-\frac{x^3}{6} + o(x^3)}{x^3 + o(x^3)} = \frac{\cancel{x^3}\left(-\frac{1}{6} + \boxed{\frac{o(x^3)}{x^3}}\right)}{\cancel{x^3}\left(1 + \boxed{\frac{o(x^3)}{x^3}}\right)} \longrightarrow -\frac{1}{6}$$

$$\boxed{\frac{o(x^3)}{x^3}} \to 0$$

$$\frac{\sin x - x + x^4}{x^3 + x^4} \overset{\boxed{n=5}}{=} \frac{\cancel{x} - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5) \cancel{-x} + x^4}{x^3 + x^4}$$

$$= \frac{-\frac{x^3}{6} + x^4 + \frac{x^5}{120} + o(x^5)}{x^3 + x^4} =$$

$$= \frac{\cancel{x^3}\left(-\frac{1}{6} + \overset{\to 0}{x} + \overset{\to 0}{\frac{x^2}{120}} + \overset{\to 0}{\boxed{\frac{o(x^5)}{x^3 x^2}}}\,\overset{\to 0}{x^2}\right)}{\cancel{x^3}\,(1 + \underset{\to 0}{x})} \to -\frac{1}{6}$$

$$\frac{\sin x - x + x^4}{x^3 + x^4} \overset{n=2}{=} \frac{\overbrace{\cancel{x} + o(x^2)}^{\sin x} - \cancel{x} + x^4}{x^3 + x^4} = \frac{o(x^2)}{x^3 + x^4}$$

$$= \frac{o(x^2)}{o(x^2)} = \text{BOH !!!}$$

Esempio 3 $\quad f(x) = \sin x + 2\cos x$. Calcolare $P_3(x)$

$$\sin x = x - \frac{x^3}{6} + o(x^3) \quad ; \quad \cos x = 1 - \frac{x^2}{2} + o(x^3)$$

$$2\cos x = 2 - x^2 + o(x^3)$$

$$\sin x + 2\cos x = x - \frac{x^3}{6} + 2 - x^2 + o(x^3)$$

$$= 2 + x - x^2 - \frac{x^3}{6} + o(x^3)$$

$$P_1(x) = 2 + x + o(x)$$

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Esempio 4 $\quad f(x) = e^x \cdot \sin x \quad$ ordine 3

$$= \underbrace{\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)\right)}_{e^x} \cdot \underbrace{\left(x - \frac{x^3}{6} + o(x^3)\right)}_{\sin x}$$

$$= x - \frac{x^3}{6} + x^2 + \frac{x^3}{2} \qquad + o(x^3)$$

$$= x + x^2 + \frac{x^3}{3} + o(x^3)$$

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Esempio 5 $\quad f(x) = \log(1+x^2)$ $\quad$ ordine 6

$$\log(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} + o(t^3)$$

$$\log(1+x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} + o(x^6)$$

$$= x^2 - \frac{x^4}{2} + \frac{x^6}{3} + o(x^7)$$

il termine successivo sarebbe x^8

— o —

$$\sin x = x - \frac{x^3}{6} + o(x^3)$$

il succ. sarebbe x^5

$$\sin x^4 = x^4 - \frac{x^{12}}{6} + o(x^{12})$$

$$+ o(x^{14})$$

$$+ o(x^{19})$$

il succ. sarebbe x^{20}