MATEMATICA I

$\boxed{\text{ORA 22}}$

Titolo nota 16/10/2007

SVILUPPINI

$\sin x = x + o(x)$

$\tan x = x + o(x)$

$\arctan x = x + o(x)$

$\arcsin x = x + o(x)$

$e^x = 1 + x + o(x)$

$\log(1+x) = x + o(x)$

$\cos x = 1 - \frac{x^2}{2} + o(x^2)$

per $x \to 0$

$\sin x = x + o(x)$ $\qquad$ $\sin x - x = o(x)$

$f(x) = \sin x - x$

$g(x) = x$

$x_0 = 0$

Verifica con def. q.e.

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{\sin x - x}{x} = \lim_{x \to 0} \left(\underbrace{\boxed{\frac{\sin x}{x}}}_{\to 1} - 1 \right) = 0$$

$\tan x = x + o(x)$, cioè $\tan x - x = o(x)$, cioè

$$\lim_{x\to 0} \frac{\tan x - x}{x} = \lim_{x\to 0} \left(\frac{\tan x}{x} - 1 \right) = 1 - 1 = 0$$

— o — o —

$e^x = 1 + x + o(x)$, cioè $e^x - 1 - x = o(x)$, cioè

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x} = \lim_{x\to 0} \left(\boxed{\frac{e^x - 1}{x}} - 1 \right) = 0$$

$\downarrow$

1

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$\log(1+x) = x + o(x)$, cioè $\log(1+x) - x = o(x)$, cioè

$$\lim_{x\to 0} \frac{\log(1+x) - x}{x} = \lim_{x\to 0} \left(\boxed{\frac{\log(1+x)}{x}} - 1 \right) = 1 - 1 = 0$$

$\downarrow$

1

$$\cos x = 1 - \frac{x^2}{2} + o(x^2), \quad \text{cioè} \quad \underbrace{\boxed{\cos x - 1 + \frac{x^2}{2}}}_{f(x)} = o(\underbrace{x^2}_{g(x)})$$

cioè

$$\lim_{x\to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^2} = \lim_{x\to 0} \left(\underbrace{\boxed{\frac{\cos x - 1}{x^2}}}_{\downarrow\, -\frac{1}{2}} + \frac{1}{2} \right) = 0$$

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Proprietà di o piccolo

$$\frac{o(g)}{g}$$

Se $f(x) = o(g(x))$ per $x \to x_0$, allora

$$\frac{f(x)}{g(x)} \to 0 \quad \text{per} \quad x \to x_0$$

(in realtà è la def. q.e.)

Esempio 1

$$\lim_{x \to 0} \frac{\sin x + \arctan x + 2 \tan x}{e^x - 1 + \log(1+x)} \qquad \left[\frac{0}{0}\right]$$

Uso gli sviluppini:

$\sin x = x + o(x)$, $\arctan x = x + o(x)$, $\tan x = x + o(x)$

$2 \tan x = 2x + 2\,o(x) = 2x + o(x)$

Numeratore: $\underbrace{x + o(x)}_{\sin x} + \underbrace{x + o(x)}_{\arctan x} + \underbrace{2x + o(x)}_{2 \tan x}$

$= 4x + o(x)$

Denominatore: $\underbrace{\cancel{1} + x + o(x)}_{e^x} \underbrace{- \cancel{1}}_{-1} + \underbrace{x + o(x)}_{\log(1+x)} = 2x + o(x)$

$$\frac{\text{Num}}{\text{Den}} = \frac{4x + o(x)}{2x + o(x)} = \frac{\cancel{x}\left(4 + \boxed{\frac{o(x)}{x}}\right)}{\cancel{x}\left(2 + \boxed{\frac{o(x)}{x}}\right)} \to \frac{4}{2} = 2$$

$\frac{o(x)}{x} \to 0$

— o — o —

Esempio 2

$$\lim_{x \to 0} \frac{e^{3x} - \cos x}{\sin(2x)}$$

$$\sin(2x) = 2x + o(2x) = 2x + o(x)$$

$$\sin t = t + o(t)$$

$$e^{3x} = 1 + 3x + o(3x) = 1 + 3x + o(x)$$

$$e^{t} = 1 + t + o(t)$$

$$\cos x = 1 - \frac{x^2}{2} + o(x^2) = 1 + o(x)$$

$x^2 = o(x)$ $\qquad -\frac{1}{2}x^2 = o(x)$ $\qquad o(x^2) = o(x)$

$$1-\frac{x^2}{2}+o(x^2)=1+o(x)+o(x)=1+o(x)$$

$$\frac{e^{3x}-\cos x}{\sin(2x)}=\frac{\cancel{1}+3x+o(x)-\cancel{1}-o(x)}{2x+o(x)}=\frac{3x+o(x)}{2x+o(x)}=$$

$$=\frac{\cancel{x}\left(3+\overbrace{\boxed{\frac{o(x)}{x}}}^{\to 0}\right)}{\cancel{x}\left(2+\underbrace{\boxed{\frac{o(x)}{x}}}_{\downarrow\, 0}\right)}\longrightarrow\frac{3}{2}$$

In alternativa:

$$\frac{e^{3x}-\cos x}{\sin(2x)}=\frac{\cancel{1}+3x+o(x)\overbrace{-\cancel{1}+\frac{x^2}{2}-o(x^2)}^{-\cos x}}{2x+o(x)}=$$

$+o-$ è lo stesso

$$=\frac{3x+\frac{x^2}{2}+o(x)+o(x^2)}{2x+o(x)}=$$

$$= \frac{\cancel{x}\left(3 + \underbrace{\boxed{\frac{x}{2}}}_{\to 0} + \underbrace{\boxed{\frac{o(x)}{x}}}_{\to 0} + \boxed{\frac{o(x^2)}{x\,x}}\boxed{x}\right)_{\to 0}}{\cancel{x}\left(2 + \underbrace{\boxed{\frac{o(x)}{x}}}_{\to 0}\right)} \longrightarrow \frac{3}{2}$$

— o — o —

__Esempio__ $\sin(3x) + \tan(2x^2) + \log(1+5x) =$

$$8x + o(x)$$

$$\sin(3x) = 3x + o(3x) = 3x + o(x)$$

$$\log(1+5x) = 5x + o(5x) = 5x + o(x)$$

$$\tan t = t + o(t) \Rightarrow \tan(2x^2) = 2x^2 + o(2x^2) = o(x)$$

$$2x^2 + o(2x^2) = 2x^2 + \omega(x)\cdot 2x^2 = x\underbrace{\left(2x + \omega(x)\cdot 2x\right)}_{\omega_1(x)} = x\,\omega_1(x)$$

$$6x + o(x) + 5x^2 - 3x^3 + o(x^4) = 6x + o(x)$$

$$\sin(\sin x) = \sin x + o(\sin x) = x + o(x) + \boxed{o(\sin x)}$$

$$\sin t = t + o(t) \qquad\qquad = x + o(x)$$

Si può fare? Cioè: $o(\sin x) = o(x)$ per $x \to 0$?

Se $f(x) = o(\sin x)$ per $x \to 0$, posso concludere che

$f(x) = o(x)$ per $x \to 0$? $\boxed{\text{SI}}$

Ipotesi $\lim_{x \to 0} \frac{f(x)}{\sin x} = 0$ Tesi: $\lim_{x \to 0} \frac{f(x)}{x} = 0$

Dim: $\frac{f(x)}{x} = \boxed{\frac{f(x)}{\sin x}}^{\nearrow 0} \cdot \boxed{\frac{\sin x}{x}}^{\nearrow 1} \to 0$

Brutalmente $f(x) = o(\sin x) \Rightarrow$ "$f(x)$ batte $\sin x$"

$\sin x$ e x "vanno a zero allo stesso modo" (pareggiano)

$\Rightarrow$ "f batte x"

— o —

Esempio 1 $\quad \dfrac{e^{x+\sin x}-1}{x} \sim \dfrac{e^{2x}-1}{2x}\,2 \longrightarrow 2$ per $x \to 0$

Più rigorosamente

$$x + \sin x = x + x + o(x) = 2x + o(x)$$

$$e^t = 1 + t + o(t) \Rightarrow e^{\boxed{2x+o(x)}} = 1 + \underset{\uparrow\, t}{\boxed{2x+o(x)}} + \underbrace{o\Big(\underset{\uparrow\, t}{\boxed{2x+o(x)}}\Big)}$$

$$= 1 + 2x + o(x)$$