Università di Pisa - Corso di Laurea in Matematica

Exam paper of "Elementi di Calcolo delle Variazioni"

Pisa, 02 February 2019

1. Let us consider the functional

$$F(u) = \int_0^1 \left(\dot{u}^2 - 3u\dot{u} + xu\right)\, dx.$$

   (a) Discuss the minimum problem for $F(u)$ with boundary conditions $u(0) = u(1) = 0$.

   (b) Discuss the minimum problem for $F(u)$ with boundary condition $u(0) = 0$.

2. Let us consider ordinary differential equation

$$\ddot{u} = \frac{(7 + \sin x)u}{7 + \sin \dot{u}}.$$

   (a) Prove that the equation admits a 2π-periodic solution.

   (b) Prove that every 4π-periodic solution is actually 2π-periodic.

3. Let us consider, for every $\ell > 0$, the problem

$$\inf\left\{\int_0^\ell \left(\sqrt{1+\dot{u}^4} - \sqrt{1+u^2}\right)\, dx : u \in C^1([0,\ell]),\ u(0) = u(\ell) = 0\right\}.$$

   (a) Determine for which values of ℓ the function $u_0(x) \equiv 0$ is a strong local minimum.

   (b) Determine for which values of ℓ the infimum is a real number.

4. Let us set

$$m_\varepsilon = \inf\left\{\int_0^1 \left(\varepsilon\dot{u}^2 + \dot{u}^4 - \sin(u^2) + u^4\right)\, dx : u \in C^1([0,1]),\ u(0) = 0,\ u(1) = \varepsilon\right\},$$

   and

$$M_\varepsilon = \inf\left\{\int_0^1 \left(\varepsilon\dot{u}^2 + \dot{u}^4 - \sin(u^{22}) + u^4\right)\, dx : u \in C^1([0,1]),\ u(0) = 0,\ u(1) = \varepsilon\right\}.$$

   For every real number $\alpha > 0$, compute the following limits:

$$\lim_{\varepsilon\to 0^+} \frac{m_\varepsilon}{\varepsilon^\alpha}, \qquad\qquad \lim_{\varepsilon\to 0^+} \frac{M_\varepsilon}{\varepsilon^\alpha}.$$

Every step has to be *suitably* motivated. Every exercise is marked considering the *correctedness* of the arguments provided and the *clarity* of the presentation. Just writing the answer without explanations deserves no marks.