1. Let us consider the functional

$$F(u) = \int_{-1}^{1} \left(\dot{u}^2 + xu + u^2\right) dx.$$

(a) Discuss the minimum problem for $F(u)$ subject to the condition $\int_{-1}^{1} u(x)\,dx = 7$.

(b) Discuss the minimum problem for $F(u)$ subject to the condition $u(0) = 7$.

(a) (ELE) $\quad (2\dot{u})' = 2u + x + \lambda \quad \rightsquigarrow \quad \ddot{u} = u + \frac{x}{2} + \lambda$

$\rightsquigarrow \quad u(x) = a\cosh x + b\sinh x - \frac{x}{2} - \lambda$

Now we need to impose the conditions

$\dot{u}(-1) = 0 \qquad \dot{u}(1) = 0 \qquad \int_{-1}^{1} u(x)\,dx = 7$

$-a\sinh(1) + b\cosh(1) - \frac{1}{2} = 0$

$a\sinh(1) + b\cosh(1) - \frac{1}{2} = 0$

$\rightsquigarrow \; a = 0, \quad b = \frac{1}{2\cosh(1)}, \quad$ last condition $\rightsquigarrow \lambda$.

This is the unique minimizer, due to the strict convexity of the Lagrangian wrt (s,p).

(b) Let us minimize $F(u)$ separately in $[-1,0]$ and $[0,1]$ with BC $u(0) = 7$. We obtain

in $[-1,0]$ $\begin{cases} \ddot{u} = u + \frac{x}{2} \\ u(0) = 7 \\ \dot{u}(-1) = 0 \end{cases} \rightsquigarrow \; u_1(x) = \underbrace{a}_{7}\cosh x + \underbrace{b}_{\frac{1}{\cosh(1)}\left(\frac{1}{2} + 7\sinh 1\right)}\sinh x + \frac{x}{2}$

in $[0,1]$ $\begin{cases} \ddot{u} = u + \frac{x}{2} \\ u(0) = 7 \\ \dot{u}(1) = 0 \end{cases} \rightsquigarrow \; u_2(x) = \underbrace{a}_{7}\cosh x + \underbrace{b}_{\frac{1}{\cosh(1)}\left(\frac{1}{2} - 7\sinh 1\right)}\sinh x + \frac{x}{2}$

Since $\dot{u}_1(0) \neq \dot{u}_2(0)$, the given pbm has no solution in C^1. The infimum is achieved by "union" of $u_1(x)$ and $u_2(x)$, which is also a minimizer in $H^1((-1,1))$.

2. Let us consider, for any value of the real parameter a, the boundary value problem

$$\ddot{u} = (x+7)(u+7), \qquad u(-7) = u(7) = a.$$

(a) Discuss existence, uniqueness and regularity of the solution.

(b) Determine the values of a for which the solution is convex.

Let us consider the minimum problem

$$\min \Big\{ \underbrace{\int_{-7}^{7} \left(\tfrac{1}{2}\dot{u}^2 + \tfrac{1}{2}(x+7)(u+7)^2\right)dx}_{F(u)} : u(-7) = u(7) = a \Big\}$$

The standard direct method works.

Since $x \geq -7$, from $F(u) \leq M$ we deduce $\int_{-7}^{7} \dot{u}^2 \leq 2M$.

Keeping the DBC into account, we obtain compactness of sublevels.

LSC is standard, as well as regularity.

Uniqueness follows from the strict convexity of the Lagrangian wrt the pair (s,p).

(b) The solution is convex if and only if $\boxed{a \geq -7.}$

- If $a < -7$, then $u(x) < -7$ in a neighborhood of $x = -7$, and therefore $\ddot{u} < 0$ in that region.

- If $a = -7$, then the solution is $u(x) \equiv -7$, which is convex

- If $a > -7$, then a truncation argument shows that $u(x) \geq -7$ for every $x \in [-7,7]$, and hence $\ddot{u} \geq 0$ in the whole interval.

Truncation argument: assume $\exists\, x_0 \in (-7,7)$ s.t. $u(x_0) < -7$.

Let $a := \min\{x \in [-7,7] : u(x) < -7\}$

$b := \max\{ \quad " \quad " \quad \}$

Then a better competitor is

$$v(x) = \begin{cases} -7 & \text{if } x \in [a,b], \\ u(x) & \text{otherwise.} \end{cases}$$

3. Let us set, for every $\ell > 0$,

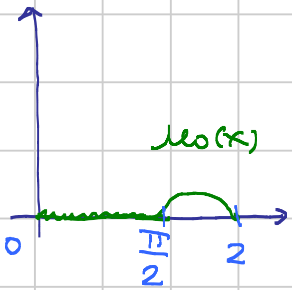
$$I(\ell) := \inf\left\{\int_0^\ell (\cos x \cdot \dot u^2 + x^2 \cdot \cos u)\,dx : u \in C^1([0,\ell]),\ u(0) = u(\ell) = 0\right\}.$$

(a) Determine the value of $I(2)$.

(b) Determine whether $I(3/2)$ is actually a minimum.

(c) Determine the value of $I(1)$.

(a) $\boxed{I(2) = -\infty}$ Let us consider any function $u_0(x)$ which vanishes in $[0, \frac{\pi}{2}] \cup \{2\}$ and is different from 0 in $(\frac{\pi}{2}, 2)$.
Then it is easy to check that $F(nu_0) \to -\infty$.
(In the same way $I(\ell) = -\infty$ for every $\ell > \frac{\pi}{2}$)



(b) $\boxed{I(\frac{3}{2}) \text{ is a minimum}}$ Indeed there exists a constant $m_0 > 0$ s.t. $\cos x \geq m_0$ for every $x \in [0, \frac{3}{2}]$

and hence

$$F(u) \geq \int_0^{3/2} (m_0 \dot u^2 + x^2 \cos u)\,dx$$

At this point the standard direct method works (key point: a bound $F(u) \leq M$ yield a bound on $\int \dot u^2 \ldots$)
(In the same way the min. exists for every $\ell < \frac{\pi}{2}$)

(c) $\boxed{I(1) = \frac{1}{3}}$ We exploit the inequalities

$\cos x \geq \cos 1 \quad \forall x \in [0,1]$, $\quad \cos u \geq 1 - \frac{1}{2}u^2 \quad \forall u \in \mathbb{R}$

and we obtain that

$$F(u) \geq \int_0^1 \left(\cos 1 \cdot \dot u^2 + x^2 - \frac{x^2}{2}u^2\right)dx \geq \underbrace{\int_0^1 x^2\,dx}_{=\frac{1}{3}} + \underbrace{\int_0^1 \left(\cos 1 \cdot \dot u^2 - \frac{1}{2}u^2\right)dx}_{Q(u) \geq 0}$$

In order to check that $Q(u) \geq 0$, it is enough to check (L^+) and (J^+), which in this case are equivalent to

$$\sqrt{\frac{1}{2\cos 1}} < \pi \iff 2\cos 1 \overset{?}{>} \frac{1}{\pi^2} \quad \text{but} \quad 2\cos 1 \underset{\substack{\uparrow \\ 1 < \frac{\pi}{3}}}{>} 2\cos\frac{\pi}{3} = 1 > \frac{1}{\pi^2}$$

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4. Let us set

$$m_\varepsilon := \inf\left\{\int_0^1 \left(\varepsilon \dot u^6 - \dot u^2 + \sin u\right)\, dx : u \in C^1([0,1]),\ u(0)=0,\ u(1)=0\right\}.$$

(a) Determine for which positive values of ε it turns out that m_ε is finite.

(b) Determine for which positive values of ε it turns out that m_ε is actually a minimum.

(c) Compute the leading term of m_ε as $\varepsilon \to 0^+$.

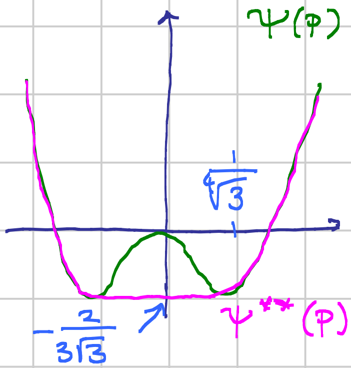
(a) m_ε is finite $\boxed{\text{for every } \varepsilon > 0}$ because the Lagrangian is bounded from below (but the bound depends on ε)

(b) m_ε is $\boxed{\text{NOT a minimum for every } \varepsilon > 0}$

Assume $u_0(x)$ is a minimum point. Due to Rolle's theorem, there exists $x_0 \in (0,1)$ s.t. $\dot u_0(x_0) = 0$. But in that point

$$L_{pp}(x_0, u_0(x_0), \dot u_0(x_0)) = -2$$

and therefore u_0 does NOT satisfy (L), and therefore it is not even a DLM.



(c) Setting $u = \frac{1}{\sqrt{\varepsilon}} v$, we obtain that

$$F(u) = \frac{1}{\sqrt{\varepsilon}} \underbrace{\int_0^1 \left(\dot v^6 - \dot v^2 + \sqrt{\varepsilon} \sin\left(\frac{v}{\sqrt{\varepsilon}}\right)\right) dx}_{G_\varepsilon(v)}$$

Now $G_\varepsilon(v) \xrightarrow{\Gamma} \underbrace{\int_0^1 \psi^{**}(\dot v)\, dx}_{G_0(v)}$, where $\psi^{**}(p)$ is the convex envelope of $\psi(p) = p^6 - p^2$

The minimum of $G_0(v)$ is $-\frac{2}{3\sqrt{3}}$, and hence $\boxed{m_\varepsilon \sim -\frac{2}{3\sqrt{3}} \frac{1}{\sqrt{\varepsilon}}}$

Checking Γ-convergence and equicoerciveness is quite standard.

Alternative, without Γ-convergence:

$$m_\varepsilon = \inf\{F_\varepsilon(u) : DBC\} = \frac{1}{\sqrt{\varepsilon}} \inf\{G_\varepsilon(v) : DBC\} = \frac{1}{\sqrt{\varepsilon}} \min\{\overline{G_\varepsilon}(v) : DBC\}$$

$$= \frac{1}{\sqrt{\varepsilon}} \min\left\{\int_0^1 \psi^{**}(\dot v) + \sqrt{\varepsilon}\sin(\dots) : DBC\right\} = \frac{1}{\sqrt{\varepsilon}}\left(-\frac{2}{3\sqrt{3}} + O(\sqrt{\varepsilon})\right).$$