3. (a) Determinare per quali esponenti α esiste una costante reale c tale che

$$\log(1+x) \le c(x-\sin x)^\alpha \qquad \forall x \ge 0.$$

(b) Determinare per quali esponenti α esiste una costante reale c tale che

$$e^{-1/x^2} \le c x^\alpha \qquad \forall x > 0.$$

(a) MODO 1

$x>0 \Rightarrow (x-\sin x)^\alpha > 0$

$$\log(1+x) \le c(x-\sin x)^\alpha \iff c \ge \frac{\log(1+x)}{(x-\sin x)^\alpha} \qquad x \ne 0$$

CONSIDERIAMO: $f(x) = \dfrac{\log(1+x)}{(x-\sin x)^\alpha} > 0$ CONTINUA $(x>0)$

COMPORTAMENTO AI LIMITI:

$$\lim_{x\to 0^+} \frac{\log(1+x)}{(x-\sin x)^\alpha} = \lim_{x\to 0^+} \frac{x+o(x)}{\left(x-x+x^3/6+o(x^3)\right)^\alpha} =$$

$$= \lim_{x\to 0^+} \frac{x}{x^{3\alpha}} \frac{1+\frac{o(x)}{x}}{\left(\frac{1}{6}+\frac{o(x^3)}{x^3}\right)^\alpha} = \begin{cases} +\infty & 3\alpha > 1 & \alpha > 1/3 \\ 6^\alpha & 3\alpha = 1 & \alpha = 1/3 \\ 0 & 3\alpha < 1 & \alpha < 1/3 \end{cases}$$

$$\lim_{x\to +\infty} \frac{\log(1+x)}{(x-\sin x)^\alpha} = \frac{\log(1+x)}{x^\alpha\left(1-\frac{\sin x}{x}\right)^\alpha} = \begin{cases} 0 & \alpha > 0 \\ +\infty & \alpha \le 0 \end{cases}$$

$\leadsto$ $0 < \alpha \le 1/3$ $f(x) > 0$ È LIMITATA SUPERIORMENTE

$$\Rightarrow \exists\, c \in \mathbb{R} \text{ t.c. } c \ge \frac{\log(1+x)}{(x-\sin x)^\alpha} \qquad \forall x > 0$$

$$x=0 \qquad \log(1+x) = 0 \le c\cdot(0-0)^\alpha = 0 \qquad \forall c \in \mathbb{R}$$

(a) MODO 2

$x>0 \Rightarrow \lg(1+x)>0 \wedge c>0$

$$\lg(1+x) \leq c\,(x-\sin x)^{\alpha} \iff \frac{1}{c} \leq \frac{(x-\sin x)^{\alpha}}{\lg(1+x)} \qquad x \neq 0$$

CONSIDERIAMO: $f(x) = \dfrac{(x-\sin x)^{\alpha}}{\lg(1+x)} > 0$ CONTINUA $(x>0)$

COMPORTAMENTO AI LIMITI:

$$\lim_{x\to 0^+} \frac{(x-\sin x)^{\alpha}}{\lg(1+x)} = \lim_{x\to 0^+} \frac{\left(x-x+x^3/6+o(x^3)\right)^{\alpha}}{x+o(x)} =$$

$$= \lim_{x\to 0^+} \frac{x^{3\alpha}}{x} \frac{\left(\frac{1}{6}+\frac{o(x^3)}{x^3}\right)^{\alpha}}{1+\frac{o(x)}{x}} = \begin{cases} 0 & 3\alpha>1 & \alpha>1/3 \\ 1/6^{\alpha} & 3\alpha=1 & \alpha=1/3 \\ +\infty & 3\alpha<1 & \alpha<1/3 \end{cases}$$

$$\lim_{x\to +\infty} \frac{(x-\sin x)^{\alpha}}{\lg(1+x)} = \frac{x^{\alpha}\left(1-\frac{\sin x}{x}\right)^{\alpha}}{\lg(1+x)} = \begin{cases} +\infty & \alpha>0 \\ 0 & \alpha\leq 0 \end{cases}$$

$\rightsquigarrow$ $0<\alpha\leq 1/3$ $f(x)>0$ È LIMITATA INFERIORMENTE

$\Rightarrow \exists\, c\in\mathbb{R}$ t.c. $\dfrac{1}{c} \leq \dfrac{(x-\sin x)^{\alpha}}{\lg(1+x)} \quad \forall x>0$

$x=0 \qquad \lg(1+x)=0 \leq c\cdot(0-0)^{\alpha}=0 \quad \forall c\in\mathbb{R}$

(b) MODO 1

$$x>0 \Rightarrow x^2>0$$

$$e^{-1/x^2} \leq c x^{\alpha} \quad \forall x>0 \iff c \geq \frac{e^{-1/x^2}}{x^{\alpha}}$$

CONSIDERIAMO: $f(x) = \frac{e^{-1/x^2}}{x^{\alpha}} > 0$ CONTINUA $(x>0)$

COMPORTAMENTO AI LIMITI:

$$\lim_{x\to 0^+} \frac{e^{-1/x^2}}{x^{\alpha}} = \lim_{x\to 0^+} \frac{1}{x^{\alpha} e^{1/x^2}} = \lim_{y\to +\infty} \frac{y^{\alpha}}{e^{y^2}} = 0 \quad \forall \alpha$$

$$\lim_{x\to +\infty} \frac{e^{-1/x^2}}{x^{\alpha}} = \lim_{x\to +\infty} \frac{1}{x^{\alpha} e^{1/x^2}} = \begin{cases} 1 & \alpha=0 \\ 0 & \alpha>0 \\ +\infty & \alpha<0 \end{cases}$$

$\rightsquigarrow$ $\alpha \geq 0$ $f(x)>0$ È LIMITATA SUPERIORMENTE

$$\Rightarrow \exists c \in \mathbb{R} \text{ t.c. } c \geq \frac{e^{-1/x^2}}{x^{\alpha}} \quad \forall x>0$$

(b) MODO 2

$$x>0 \Rightarrow x^{\alpha}>0 \wedge c>0$$

$$e^{-1/x^2} \leq c x^{\alpha} \quad \forall x>0 \iff \frac{1}{c} \leq \frac{x^{\alpha}}{e^{-1/x^2}}$$

CONSIDERIAMO: $f(x) = \frac{x^{\alpha}}{e^{-1/x^2}} > 0$ CONTINUA $(x>0)$

COMPORTAMENTO AI LIMITI:

$$\lim_{x\to 0^+} \frac{x^{\alpha}}{e^{-1/x^2}} = \lim_{x\to 0^+} x^{\alpha} e^{1/x^2} = \lim_{y\to +\infty} \frac{e^{y^2}}{y^{\alpha}} = +\infty \quad \forall \alpha$$

$$\lim_{x\to +\infty} \frac{x^{\alpha}}{e^{-1/x^2}} = \lim_{x\to +\infty} x^{\alpha} e^{1/x^2} = \begin{cases} 1 & \alpha=0 \\ +\infty & \alpha>0 \\ 0 & \alpha<0 \end{cases}$$

$\rightsquigarrow$ $\alpha \geq 0$ $f(x)>0$ È LIMITATA INFERIORMENTE

$$\Rightarrow \exists c \in \mathbb{R} \text{ t.c. } \frac{1}{c} \leq \frac{x^{\alpha}}{e^{-1/x^2}} \quad \forall x>0$$