Integrali 6

Argomenti: Tecniche di integrazione **Difficoltà**: ⋆⋆⋆⋆

Prerequisiti: Sostituzioni razionalizzanti

Determinare una primitiva delle seguenti funzioni (e fare la verifica).

	Funzione	Primitiva (a)	Funzione	Primitiva (b)
1	$\frac{1}{1+e^x}$	$x-\log(1+e^x)$	$\frac{1}{1+e^{4x}}$	$x-\log\sqrt[4]{e^{4x}+1}$
2	$\frac{e^x+1}{e^x-1}$	$2\log\lvert e^x-1\rvert - x$	$\frac{e^{3x}}{e^{2x}+e^x}$	$e^x-\log(1+e^x)$
3	$\frac{4^x}{2^x+1}$	$\frac{1}{\log 2}\left(2^x-\log(2^x+1)\right)$	$\frac{2^x}{4^x+1}$	$\frac{1}{\log 2}\arctan(2^x)$
4	$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x$	$\frac{1}{\sqrt{2-x^2}}$	$\arcsin\left(\frac{x}{\sqrt{2}}\right)$
5	$\frac{1}{\sqrt{x^2+1}}$	$-\log\left(\sqrt{x^2+1}-x\right)$	$\frac{1}{\sqrt{x^2-1}}$	$-\log\lvert\sqrt{x^2-1}-x\rvert$
6	$\frac{x^3}{\sqrt{x^4+1}}$	$\frac{1}{2}\sqrt{x^4+1}$	$\frac{x}{\sqrt{x^4+1}}$	$-\frac{1}{2}\log\left(\sqrt{x^4+1}-x^2\right)$
7	$\arcsin x$	$x\cdot\arcsin x+\sqrt{1-x^2}$	$\arccos x$	$x\cdot\arccos x-\sqrt{1-x^2}$
8	$\sqrt{1-x^2}$	$\frac{1}{2}x\sqrt{1-x^2}+\frac{1}{2}\arcsin x$	$\sqrt{2-x^2}$	$\frac{x}{2}\sqrt{2-x^2}+\arcsin\frac{x}{\sqrt{2}}$
9	$\sqrt{3-2x^2}$	$\frac{x}{2}\sqrt{3-2x^2}+\frac{3}{2\sqrt{2}}\arcsin\left(\frac{\sqrt{2}\,x}{\sqrt{3}}\right)$	$x\arcsin x$	$\frac{2x^2-1}{4}\arcsin x+\frac{x}{4}\sqrt{1-x^2}$
10	$\sqrt{x^2+1}$	$\frac{x}{2}\sqrt{x^2+1}+\frac{1}{2}\log\left(\sqrt{x^2+1}+x\right)$	$\sqrt{x^2-1}$	$\frac{x}{2}\sqrt{x^2-1}-\frac{1}{2}\log\lvert x+\sqrt{x^2-1}\rvert$
11	$x\sqrt{x+1}$	$\frac{2}{5}(x+1)^{5/2}-\frac{2}{3}(x+1)^{3/2}$	$x\sqrt{x^2-1}$	$\frac{1}{3}(x^2-1)^{3/2}$
12	$\frac{x}{\sqrt{x+1}}$	$\frac{2}{3}(x+1)^{3/2}-2(x+1)^{1/2}$	$\frac{\sqrt{x+1}}{x}$	$2\sqrt{x+1}+\log\frac{\lvert x\rvert}{(\sqrt{x+1}+1)^2}$
13	$\sqrt{\frac{x}{x+1}}$	$\sqrt{x(x+1)}+\frac{1}{2}\log\left\lvert\frac{\sqrt{\frac{x}{x+1}}-1}{\sqrt{\frac{x}{x+1}}+1}\right\rvert$	$\frac{x^3}{\sqrt{x^2+1}}$	$\frac{(x^2+1)^{3/2}}{3}-(x^2+1)^{1/2}$
14	$\frac{x+\sqrt{x+1}}{\sqrt{x}}$	$\frac{2}{3}x^{3/2}+\sqrt{x(x+1)}-\frac{1}{2}\log\frac{\sqrt{\cdots}-1}{\sqrt{\cdots}+1}$	$\sqrt[3]{1+\sqrt{x}}$	$\frac{6}{7}(1+\sqrt{x})^{7/3}-\frac{3}{2}(1+\sqrt{x})^{4/3}$
15	$\frac{\sqrt{x}}{\sqrt[3]{x}+1}$	$6\left(\frac{x^{7/6}}{7}-\frac{x^{5/6}}{5}+\frac{\sqrt{x}}{3}-\sqrt[6]{x}+\text{arct.}\sqrt[6]{x}\right)$	$\frac{1}{\sin x}$	$\frac{1}{2}\log\lvert\cos x-1\rvert-\frac{1}{2}\log\lvert\cos x+1\rvert$
16	$\frac{\tan^{-1}x}{(1+\sin^2 x)^{3/2}}$	$\frac{1}{2}\log\left\lvert\frac{\sqrt{1+\sin^2 x}-1}{\sqrt{1+\sin^2 x}+1}\right\rvert+\frac{1}{\sqrt{1+\sin^2 x}}$	$\frac{1}{1+\sin x}$	$-\frac{2}{1+\tan\left(\frac{x}{2}\right)}$

INTEGRALI 6 Esercizi A.M I Parte A Test 41

1a) $\int \frac{1}{1+e^x}\,dx \quad y=e^x \quad dy=e^x\,dx \qquad \int \frac{1}{1+e^x}\cdot\frac{e^x}{e^x}\,dx =$

$\int\left(\frac{1}{1+y}\cdot\frac{1}{y}\right)dy = \int \frac{1}{y(1+y)}\,dy = \frac{A}{y}+\frac{B}{1+y} \qquad A+B=0 \quad A=1 \quad B=-1$

$\int\left(\frac{1}{y}-\frac{1}{1+y}\right)dy = \log e^x - \log(1+e^x) = \boxed{x-\log(1+e^x)}$

1b) $\int \frac{1}{1+e^{4x}}\,dx \quad e^{4x}=y \quad dy=4e^{4x}\,dx$

$\int \frac{1}{1+e^{4x}}\cdot\frac{e^x}{e^x}\,dx = \frac{1}{4}\int \frac{1}{1+y}\cdot\frac{1}{y}\,dy$ come 1a

$=\frac{1}{4}\left(\log e^{4x} - \log(e^x+1)\right) = \frac{1}{4}\log e^{4x} - \frac{1}{4}\log(e^{4x}+1)$

$= \boxed{x-\log\sqrt[4]{e^{4x}+1}}$

2a) $\int \frac{e^x+1}{e^x-1}\,dx \quad e^x=y \quad dy=e^x\,dx \qquad \int \frac{e^x+1}{e^x-1}\cdot\frac{e^x}{e^x}\,dx$

$\int \frac{y+1}{y-1}\cdot\frac{1}{y}\,dy \qquad \frac{y+1}{y(y-1)} = \frac{A}{y}+\frac{B}{y-1} = \frac{Ay-A+By}{y(y-1)}$

$x^1)$ $\left\{\begin{array}{ll} A+B=1 & B=2 \\ -A=1 & A=-1 \end{array}\right.$ t.n) $\qquad -\int\frac{1}{y}\,dy + 2\int\frac{1}{y-1}\,dy = -\log|y| + 2\log|y-1|$

$= \boxed{2\log|e^x-1| - x}$

2b) $\int \frac{e^{3x}}{e^{2x}+e^{x}}\,dx = \int \frac{e^{2x}}{e^{2x}+e^{x}}\, e^{x}\,dx \qquad e^{x}=y \qquad = dy = e^{x}dx$

$\int \frac{y^2}{y^2+y}\,dy = \int \frac{y^{\cancel{2}}}{\cancel{y}(y+1)}\,dy = \int \frac{y+1}{y+1}\,dy - \int \frac{1}{y+1}\,dy = \int dy - \int \frac{1}{y+1}\,dy$

$= y - \log|y+1| \qquad = \boxed{e^{x} - \log(1+e^{x})}$

3a) $\int \frac{4^{x}}{2^{x}+1}\,dx = \int \frac{2^{2x}}{2^{x}+1}\,dx \qquad t = 2^{x} \quad dt = \log 2 \cdot 2^{x}\,dx$

$\frac{1}{\log 2}\int \frac{t}{t+1}\,dt = \frac{1}{\log 2}\left(t - \log|t+1|\right) \qquad \boxed{= \frac{1}{\log 2}\left(2^{x} - \log(2^{x}+1)\right)}$

come esercizio prima

3b) $\int \frac{2^{x}}{4^{x}+1}\,dx = \int \frac{2^{x}}{2^{2x}+1}\,dx \qquad 2^{x}=t \quad dt = 2^{x}\log 2\,dx$

$= \frac{1}{\log 2}\int \frac{1}{1+t^2}\,dt = \frac{1}{\log 2}\arctan(t) \qquad = \boxed{\frac{1}{\log 2}\arctan(2^{x})}$

4a) $\int \frac{1}{\sqrt{1-x^2}}\,dx = \boxed{\arcsin(x)} \qquad x = \sin t \quad dx = \cos t\,dt$

$\int \frac{\cos t}{\sqrt{1-\sin^2 t}}\,dt = \int \frac{\cos t}{\sqrt{\cos^2 t}}\,dt = \int dt = t \qquad x = \sin t \qquad t = \arcsin(x)$

4b) $\int \frac{1}{\sqrt{2-x^2}}\,dx = \int \frac{\sqrt{2}}{\sqrt{2-2y^2}}\,dy = \int \frac{1}{\sqrt{1-y^2}}\,dy = \arcsin(y) = \boxed{\arcsin\left(\frac{x}{\sqrt{2}}\right)}$

$y = \frac{x}{\sqrt{2}} \quad dy = \frac{1}{\sqrt{2}}\,dx \quad dx = \sqrt{2}\,dy \qquad x = \sqrt{2}y$

5a) $\int \frac{1}{\sqrt{x^2+1}}\,dx = \boxed{\text{Settsinh}\, x = \log\left(\sqrt{x^2+1}+x\right) = -\log\left(\sqrt{x^2+1}-x\right)}$

1°) Pongo $x = \sinh(t)$ $\quad dx = \cosh(t)\,dt$

$$\int \frac{\cosh t}{\sqrt{\sinh^2 t+1}}\,dt = t = \text{settosinh}(x)$$

2°) Pongo $(y+x)^2 = \left(\sqrt{x^2+1}\right)^2$ $\qquad x = \frac{1-y^2}{2y}$

$$dx = \left[\frac{1-y^2}{2y}\right]' dy \quad dx = \left[-\frac{y^2+1}{2y^2}\right] dy \quad ; \quad y+x = \boxed{y+\frac{1-y^2}{2y}}$$

$$\int \boxed{\frac{\cancel{2y}}{\cancel{1+y^2}}} \cdot \left(-\frac{\cancel{1+y^2}}{\cancel{2}y^2}\right) dy = -\int \frac{1}{y}\,dy = -\log|y| = -\log\left(\sqrt{x^2+1}-x\right)^{-1}$$

$A^2+B^2 = (A+B)(A-B)$

$$= \log\left(\sqrt{x^2+1}+x\right)$$

3° Pongo $y = \sqrt{1+x^2}-x$ $\qquad dy = \left(\frac{x}{\sqrt{1+x^2}}-1\right)dx = \frac{x-\sqrt{1+x^2}}{\sqrt{1+x^2}}\,dx$

cioè $\frac{1}{\sqrt{1+x^2}}\,dx = -\frac{1}{y}\,dy$ poi come prima.

5B

$\int \frac{1}{\sqrt{x^2-1}}\,dx$ $\quad$ pongo $y+x = \sqrt{x^2-1} \implies x = -\frac{y^2+1}{2y}$

$$dx\left[-\frac{y^2+1}{2y}\right]' dy = -\frac{y^2-1}{2y^2}\,dy$$

$$\int \frac{1}{y+x} [redacted] = \int \frac{1}{y-\frac{y^2+1}{2y}} \cdot \left(-\frac{y^2-1}{2y^2}\right) dy =$$

$$= \int \frac{\cancel{2y}}{\cancel{y^2-1}} \cdot \left(- \frac{\cancel{y^2-1}}{2y^2} \right) dy = - \int \frac{1}{y} dy = - \log|y|$$

$$y + x = \sqrt{x^2-1} \qquad y = \sqrt{x^2-1} - x$$

$$\int \frac{1}{\sqrt{x^2-1}} dx = \boxed{- \log\left|\sqrt{x^2-1} - x\right|}$$

6a) $$\int \frac{x^3}{\sqrt{x^4+1}} dx \qquad y = x^4+1 \qquad dy = 4x^3 dx$$

$$\frac{1}{4} \int \frac{dy}{\sqrt{y}} = \frac{1}{4} \cdot 2\sqrt{y} = \frac{1}{2}\sqrt{y} \qquad = \boxed{\frac{1}{2}\sqrt{x^4+1}}$$

6b) $$\int \frac{x}{\sqrt{x^4+1}} dx \qquad y = x^2 \quad dy = 2x\, dx$$

$$\frac{1}{2} \int \frac{1}{\sqrt{y^2+1}} dy$$ (già fatto vedere 5a)

$$= -\frac{1}{2} \log\left(\sqrt{y^2+1} - y\right) \qquad = \boxed{-\frac{1}{2} \log\left(\sqrt{x^4+1} - x^2\right)}$$

7a) $\int \arcsin x \, dx$ (per parti con 1 nascosto) $\int \underset{g}{1} \cdot \underset{F}{\arcsin x} \, dx =$

$$\int \underset{g}{1} \cdot \underset{F}{\arcsin x} \, dx = \underset{G}{x} \, \underset{F}{\arcsin x} \boxed{- \int \underset{f}{\frac{1}{\sqrt{1-x^2}}} \cdot \underset{G}{x} \, dx}$$

$$y = 1 - x^2 \qquad dy = -2x\, dx$$

$$+\frac{1}{2}\int \frac{1}{\sqrt{y}}\,dy = \frac{1}{2}\cdot 2\sqrt{y} = \sqrt{1-x^2}$$

$$\int \arcsin x\,dx = \boxed{x\cdot\arcsin x + \sqrt{1-x^2}}$$

7B)

$$\int \arccos x\,dx = \int 1\,\arccos x\,dx = x\arccos x - \int -\frac{1}{\sqrt{1-x^2}}\,x\,dx$$

$$y = 1-x^2 \quad dy = -2x\,dx; \quad \Rightarrow -\frac{1}{2}\int \frac{1}{\sqrt{y}}\,dy = -\sqrt{y} = -\sqrt{1-x^2}$$

$$\int \arccos x\,dx = \boxed{x\cdot\arccos x - \sqrt{1-x^2}}$$

8A)

$$\int \sqrt{1-x^2}\,dx \qquad x = \sin(t) \quad dx = \cos t\,dt$$

$$\int \sqrt{1-\sin^2 t}\cdot\cos t\,dt = \int \cos^2 t\,dt \qquad \cos^2 t = \frac{\cos 2t + 1}{2}$$

$$= \frac{1}{2}\int \cos 2t + 1\,dt = \frac{1}{4}\sin 2t + \frac{t}{2} \qquad \sin 2t = 2\sin t\cos t$$

$$= \frac{1}{2}\sin t\sqrt{1-\sin^2 t} + \frac{t}{2} \qquad \left[\begin{array}{l} t = \arcsin(x) \\ x = \sin t \end{array}\right]$$

$$= \boxed{\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\arcsin x}$$

8b) $\int \sqrt{2-x^2}\, dx \qquad x = \sqrt{2}\sin t \qquad dx = \sqrt{2}\cos t \, dt$

$$\int \sqrt{2-(\sqrt{2}\sin t)^2}\cdot \sqrt{2}\cos t \, dt = 2\int \cos^2 dt$$

$$= \frac{2}{2}\int \cos 2t + 1 \;\, dt = \frac{1}{2}\sin 2t + t = \frac{1}{2}\, 2\sin t \cos t + t$$

$$= \sin t \sqrt{1-\sin^2 t} + t \qquad t = \arcsin \frac{x}{\sqrt{2}}$$

$$\sin t = \frac{x}{\sqrt{2}}$$

$$= \frac{x}{\sqrt{2}}\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2} + \arcsin \frac{x}{\sqrt{2}}$$

$$= \boxed{\frac{x}{2}\sqrt{2-x^2} + \arcsin \frac{x}{\sqrt{2}}}$$

9a $\int \sqrt{3-2x^2}\, dx \qquad x = \frac{\sqrt{3}}{\sqrt{2}}\sin t \qquad \sin t = \frac{\sqrt{2}\,x}{\sqrt{3}} \qquad dx = \frac{\sqrt{3}}{\sqrt{2}}\cos t$

$$t = \arcsin\left(\frac{\sqrt{2}\,x}{\sqrt{3}}\right)$$

$$\int \sqrt{3-2\left(\frac{\sqrt{3}}{\sqrt{2}}\sin t\right)^2}\cdot \frac{\sqrt{3}}{\sqrt{2}}\cos t \, dt = \int \sqrt{\underbrace{3-3\sin^2 t}_{3(1-\sin^2 t)}}\cdot \frac{\sqrt{3}}{\sqrt{2}}\cos t \, dt =$$

$$\frac{3}{\sqrt{2}}\int \cos^2 t \, dt = \frac{3}{\sqrt{2}}\left(\underbrace{\frac{1}{2}\sin t\sqrt{1-\sin^2 t} + \frac{t}{2}}_{\underline{8a}}\right)$$

$$= \frac{\cancel{3}}{2\cancel{\sqrt{2}}}\left(\frac{\cancel{\sqrt{2}}\,x}{\cancel{\sqrt{3}}}\cdot\sqrt{1-\frac{2}{\cancel{3}}x^2}\right) + \frac{3}{2\sqrt{2}}\arcsin \frac{\sqrt{2}\,x}{\sqrt{3}}$$

$$= \boxed{\frac{1}{2}x\sqrt{3-2x^2} + \frac{3}{2\sqrt{2}}\arcsin\left(\frac{\sqrt{2}\,x}{\sqrt{3}}\right)}$$

9b) $$\int x \arcsin x \, dx \underset{\text{per parti}}{=} \frac{x^2}{2}\arcsin x - \int \frac{1}{\sqrt{1-x^2}} \cdot \frac{x^2}{2} dx$$

$$= \frac{x^2}{2}\arcsin x \boxed{-\frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}} dx} \qquad x = \sin t \qquad t = \arcsin x$$

$$-\frac{1}{2}\int \frac{\sin^2 t}{\cos t} \cdot \cos t \, dt = -\frac{1}{2}\int \sin^2 t \, dt = -\frac{1}{2}\underbrace{\left(\frac{t}{2} - \frac{1}{2}\sin t \cos t\right)}_{\text{esercizio 4a INTEGRALI 3}}$$

$$= -\frac{1}{2}\left(\frac{t}{2} - \frac{1}{2}\sin t \sqrt{1-\sin^2 t}\right) = \boxed{-\frac{1}{2}\left(\frac{1}{2}\arcsin x - \frac{1}{2} x\sqrt{1-x^2}\right)} =$$

$$\int x \arcsin x \, dx = \frac{x^2}{2}\arcsin x - \frac{1}{4}\arcsin x + \frac{1}{4} x \sqrt{1-x^2}$$

$$= \boxed{\frac{2x^2-1}{4} \cdot \arcsin x + \frac{1}{4} x\sqrt{1-x^2}}$$

10a) $$\int \sqrt{x^2+1} \, dx = \int \underset{g}{1} \underset{F}{\sqrt{x^2+1}} \, dx = x \cdot \sqrt{x^2+1} \boxed{- \int \frac{x}{\sqrt{x^2+1}} \cdot x \, dx}$$

$$\boxed{- \int \frac{x^2+1}{\sqrt{x^2+1}} - \frac{1}{\sqrt{x^2+1}} dx = -\int \sqrt{x^2+1} - \frac{1}{\sqrt{x^2+1}} dx}$$ GRANDE RITORNO

$$2\int \sqrt{x^2+1} \, dx = x\sqrt{x^2+1} + \underbrace{\int \frac{1}{\sqrt{x^2+1}} dx}_{\text{esercizio 5a}} =$$

$$\int \sqrt{x^2+1} \, dx = \boxed{\frac{1}{2} x \sqrt{x^2+1} + \frac{1}{2}\log\left(\sqrt{x^2+1} + x\right)}$$

10b) $\int \sqrt{x^2-1}\, dx \qquad x = \cosh t \qquad t = \text{settcosh}\, x = \log\left(x+\sqrt{x^2-1}\right)$

$x \in [1, +\infty)$

$dx = \sinh t\, dt \qquad \cosh^2 t + \sinh^2 t = 1$

$\int \sqrt{\cosh^2 t - 1} \cdot \sinh t\, dt = \int \sinh^2 t\, dt$ per parti.

$\int \sinh t \cdot \sinh t\, dt = \sinh t \cosh t - \int \cosh^2 t\, dt$

$= \sinh t \cosh t - \int (\sinh^2 t + 1)\, dt$

$= \sinh t \cosh - t - \int \sinh^2 t\, dt$ grande ritorno

$\int \sinh^2 t\, dt = \frac{1}{2}\left(\sinh t \cosh t - t\right) = \frac{1}{2}\left(\sqrt{\cosh^2 t - 1} \cdot \cosh t - t\right)$

$\int \sqrt{x^2-1}\, dx = \boxed{\frac{1}{2} x \sqrt{x^2-1} - \frac{1}{2} \log\left|x+\sqrt{x^2-1}\right|}$

11a) $\int x\sqrt{x+1}\, dx \qquad x+1 = y \quad dy = dx \quad ; \quad x = y-1$

$\int (y-1)\sqrt{y}\, dy = \int \left(y^{3/2} - y^{1/2}\right) dy = \frac{2}{5} y^{5/2} - \frac{2}{3} y^{3/2}$

$= \boxed{\frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2}}$

11b) $\int x\sqrt{x^2-1}\, dx \quad ; \quad x^2 - 1 = y \qquad dy = 2x\, dx$

$\frac{1}{2}\int \sqrt{y}\, dy = \frac{1}{2} \cdot \frac{2}{3} y^{3/2} = \boxed{\frac{1}{3}\left(x^2-1\right)^{3/2}}$

12a) $\displaystyle\int \frac{x}{\sqrt{x+1}}\,dx \qquad y = x+1 \qquad dy = dx \qquad x = y-1$

$$\int \frac{y-1}{\sqrt{y}}\,dy = \int\left(\sqrt{y} - \frac{1}{\sqrt{y}}\right)dy = \frac{2}{3}y^{3/2} - 2y^{1/2} = \boxed{\frac{2}{3}(x+1)^{3/2} - 2(x+1)^{1/2}}$$

12b) $\displaystyle\int \frac{\sqrt{x+1}}{x}\,dx \qquad y = \sqrt{x+1} \qquad y^2 = x+1 \qquad x = y^2-1$

$dx = 2y\,dy$

$$\int \frac{\sqrt{x+1}}{x}\,dx = 2\int \frac{y^2+1-1}{y^2-1}\,dy = 2\int \frac{y^2-1}{y^2-1}\,dy + \int \frac{1}{y^2-1}\,dy$$

$$= 2y + 2\int \frac{1}{(y-1)(y+1)}\,dy \;;\quad \frac{A}{(y-1)} + \frac{B}{y+1} = \qquad A = \frac{1}{2}$$

$$B = -\frac{1}{2}$$

$$= 2y + \boxed{2\int \frac{1}{2}\frac{1}{y-1} - \frac{1}{2}\frac{1}{y+1}\,dy = \log\left|\frac{y-1}{y+1}\right|}$$

$$\int \frac{\sqrt{x+1}}{x}\,dx = 2\sqrt{x+1} + \log\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right|$$

$$= \boxed{2\sqrt{x+1} + \log\frac{|x|}{\left(\sqrt{x+1}+1\right)^2}}$$

13a) $\displaystyle\int \sqrt{\frac{x}{x+1}}\,dx \qquad y = \sqrt{\frac{x}{x+1}} \qquad y^2 = \frac{x}{x+1} \qquad x = -\frac{y^2}{y^2-1}$

$$\int \underset{F}{y}\underset{g}{\left[-\frac{y^2}{y^2-1}\right]'}dy = \underset{F}{y}\cdot\underset{G}{\left(-\frac{y^2}{y^2-1}\right)} \boxed{- \int \underset{f}{1}\cdot\left(\frac{-y^2}{y^2-1}\right)} dy$$

$$\boxed{+\int \frac{y^2-1+1}{y^2-1}\,dy} = \int 1 + \frac{1}{(y-1)(y+1)}\,dy = y + \frac{1}{2}\log\left|\frac{y-1}{y+1}\right|$$

$$= y\cdot\left(-\frac{y^2}{y^2-1}\right) + y + \frac{1}{2}\log\left|\frac{y-1}{y+1}\right|$$ tornando in x

$$\boxed{= \sqrt{x(x+1)} + \frac{1}{2}\log\left|\frac{\sqrt{\frac{x}{x+1}}-1}{\sqrt{\frac{x}{x+1}}+1}\right|}$$

13b) $\int \frac{x^3}{\sqrt{x^2+1}}\,dx$ $\quad y = x^2+1 \quad x^2 = y-1$

$dy = 2x\,dx$

$$\frac{1}{2}\int \frac{y-1}{\sqrt{y}}\,dy = \frac{1}{2}\int \left(\sqrt{y} - \frac{1}{\sqrt{y}}\right) dy = \frac{1}{2}\frac{2}{3}y^{3/2} - \frac{1}{2}\cdot 2y^{1/2}$$

$$= \boxed{\frac{(x^2+1)^{3/2}}{3} - (x^2+1)^{1/2}}$$

14a) $\int \frac{x+\sqrt{x+1}}{\sqrt{x}}\,dx = \int \left(\sqrt{x} + \sqrt{\frac{x+1}{x}}\,dx\right)$ $\quad \boxed{\int \sqrt{x}\,dx = \frac{2}{3}x^{3/2}}$

$$\int \sqrt{\frac{x+1}{x}}\,dx \qquad y = \sqrt{\frac{x+1}{x}} \qquad y^2 = \frac{x+1}{x} \qquad x = \frac{1}{y^2-1}$$

$$\int y\left[\frac{1}{y^2-1}\right]' dy = y\cdot\frac{1}{y^2-1} - \int 1\cdot\frac{1}{y^2-1}\,dy$$

$$= \frac{y}{y^2-1} - \int \frac{1}{(y-1)(y+1)}\,dy = \sqrt{x(x+1)} - \frac{1}{2}\log\left(\frac{\sqrt{\frac{x+1}{x}}-1}{\sqrt{\frac{x+1}{x}}+1}\right)$$

$$\int \frac{x+\sqrt{x+1}}{\sqrt{x}}\,dx = \boxed{\frac{2}{3}x^{3/2} + \sqrt{x(x+1)} - \frac{1}{2}\log\left(\frac{\sqrt{\frac{x+1}{x}}-1}{\sqrt{\frac{x+1}{x}}+1}\right)}$$

14b) $\int \sqrt[3]{1+\sqrt{x}}\,dx \qquad y = 1+\sqrt{x} \qquad \frac{dy}{dx} = \frac{1}{2\sqrt{x}} \qquad dx = 2\sqrt{x}\,dy$

$\sqrt{x} = y-1$

$$2\int \sqrt[3]{y}\cdot(y-1)\,dy = 2\int\left(y^{4/3} - y^{1/3}\right)dy.$$

$$= 2\cdot\frac{3y^{7/3}}{7} - 2\,\frac{3y^{4/3}}{4} = \boxed{\frac{6}{7}\left(1+\sqrt{x}\right)^{7/3} - \frac{3}{2}\left(1+\sqrt{x}\right)^{4/3}}$$

15a) $\frac{\sqrt{x}}{\sqrt[3]{x}+1}\,dx \qquad t = \sqrt[6]{x} \qquad x = t^6 \qquad dx = 6t^5\,dt$

$$\begin{array}{l|l} t^8 & t^2+1 \\ \hline -t^8 - t^6 & t^6-t^4+t^2-1 \\ \hline = \;\; -t^6 & \\ \;\;\; +t^6 + t^4 & \\ \hline = \;\; t^4 - t^2 & \\ \;\;\; +t^2 + 1 & \\ \hline = \;\; 1 & \end{array}$$

$$6\int \frac{t^3}{t^2+1}\,t^5\,dt = 6\int \frac{t^8}{t^2+1}\,dt$$

$$= 6\int \frac{(t^2+1)(t^6-t^4+t^2-1)+1}{(t^2+1)}\,dt$$

$$t^8 = (t^2+1)(t^6-t^4+t^2-1)+1$$

$$= 6\int (t^6-t^4+t^2-1)\,dt + 6\int \frac{1}{t^2+1}\,dt =$$

$$= 6\left(\frac{t^7}{7} - \frac{t^5}{5} + \frac{t^3}{3} - t\right) + 6\arctan(t) =$$

$$= \boxed{6\left(\frac{x^{7/6}}{7} - \frac{x^{5/6}}{5} + \frac{\sqrt{x}}{3} - \sqrt[6]{x}\right) + 6\arctan\sqrt[6]{x}}$$

15b) $$\int \frac{1}{\sin x}\,dx = \int \frac{\sin x}{\sin^2 x}\,dx = \int \frac{\sin x}{1-\cos^2 x}\,dx$$

$y = \cos x \quad dy = -\sin x\,dx$ $$= \int \frac{-dy}{1-y^2} = \int \frac{dy}{y^2-1}$$

$$\frac{1}{y^2-1} = \frac{A}{y-1} + \frac{B}{y+1} \qquad A = 1/2 \quad B = -1/2$$

$$\frac{1}{2}\int \frac{1}{y-1} - \frac{1}{y+1}\,dy = \frac{1}{2}\log|y-1| - \frac{1}{2}\log|y+1|$$

$$= \boxed{\frac{1}{2}\log|\cos x - 1| - \frac{1}{2}\log|\cos x + 1|}$$

16a) $$\int \frac{\tan^{-1} x}{(1+\sin^2 x)^{3/2}}\,dx \qquad \sin x = y \quad dy = \cos x\,dx \qquad \int \frac{\cos x}{\sin x\,(1+\sin^2 x)^{3/2}}\,dx$$

$$= \int \frac{dy}{y\,(1+y^2)^{3/2}} \qquad \sqrt{1+y^2} = t \qquad y^2 = t^2 - 1$$

$$\frac{dt}{dy} = \frac{y}{\sqrt{y^2+1}} \qquad dy = \frac{\sqrt{y^2+1}}{y}\,dt$$

dy

$$\int \frac{\cancel{\sqrt{y^2+1}}}{y \cdot y \cdot \sqrt{(y^2+1)^{\cancel{3/2}}}} = \int \frac{dt}{(t^2-1)\,t^2}$$

$$y^2 \qquad \sqrt{(y^2-1)^2}$$

$$\frac{1}{t^2(t-1)(t+1)} = \frac{A}{t} + \frac{B}{t-1} + \frac{C}{t+1} + \frac{d}{dt}\frac{D}{t}$$

$$\frac{A}{t} + \frac{B}{t-1} + \frac{C}{t+1} - \frac{D}{t^2} = \frac{At(t-1)(t+1) + Bt^2(t+1) + Ct^2(t-1) - D(t^2-1)}{(t^2-1)\,t^2}$$

numeratore $\quad At^3 - At + Bt^3 + Bt^2 + Ct^3 - Ct^2 - Dt^2 + D$

$t^3)\quad A+B+C=0 \qquad B=-C \qquad B=\frac{1}{2}$

$t^2)\quad B-C-D=0 \qquad -2C=1 \qquad C=-\frac{1}{2}$

$t)\quad A=0$

$t^0)\quad D=1$

$$\frac{1}{t^2(t^2-1)} = \frac{\frac{1}{2}}{t-1} + \frac{-\frac{1}{2}}{t+1} + \frac{d}{dt}\cdot\frac{1}{t}$$

$$\int \frac{1}{t^2(t^2-1)}\,dt = \frac{1}{2}\int\frac{1}{t-1}\,dt - \frac{1}{2}\int\frac{1}{t+1}\,dt + \int\frac{d}{dt}\cdot\frac{1}{t}\,dt$$

$$= \frac{1}{2}\log|t-1| - \frac{1}{2}\log|t+1| + \frac{1}{t}$$

tornando in x

$t = \sqrt{1+y^2}$ e $y = \sin x$

$$\boxed{\frac{1}{2}\log\left|\frac{\sqrt{1+\sin^2 x}-1}{\sqrt{1+\sin^2 x}+1}\right| + \frac{1}{\sqrt{1+\sin^2 x}}}$$

16 b)

$$\int \frac{1}{1+\sin x} dx \qquad \Bigg| \qquad \sin x = \frac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}$$

$$y = \tan\left(\frac{x}{2}\right) \qquad dy = \left(1+\tan^2\left(\frac{x}{2}\right)\right)\cdot\frac{1}{2}\,dx \qquad dx = \frac{2y}{1+y}$$

$$\int \frac{1}{1+\sin x} dx = \int \frac{1}{1+\frac{2y}{1+y^2}} \cdot \frac{2\,dy}{1+y^2} = 2\int \frac{1}{y^2+2y+1} dy$$

$$= 2\int \frac{1}{(y+1)^2} dy = -\frac{2}{1+y} \Rightarrow \boxed{-\frac{2}{1+\tan\left(\frac{x}{2}\right)}}$$

FINE..