

ESERCIZI SU INTEGRAZI DOPPI

• $D = \{ [0,1] \times [0,1] \}$

$f(x,y) = x+y$

$$\int_0^1 \int_0^1 x+y = \int_0^1 dx \int_0^1 x+y dy = \int_0^1 \left[xy + \frac{y^2}{2} \right]_0^1 dx = \int_0^1 \left(x + \frac{1}{2} \right) dx$$

$$= \left[\frac{x^2}{2} + \frac{1}{2}x \right]_0^1 = \frac{1}{2} + \frac{1}{2} = 1$$

• $D = \{ [1,2] \times [0,3] \}$

$$\int_1^2 \int_0^3 x+y dx dy = \int_1^2 dx \int_0^3 (x+y) dy = \int_1^2 dx \left[xy + \frac{y^2}{2} \right]_0^3 =$$

$$= \int_1^2 \left[3x + \frac{9}{2} \right] dx = \left[\frac{3x^2}{2} + \frac{9}{2}x \right]_1^2 = 6 + 9 - \frac{3}{2} - \frac{9}{2} =$$

$$= \frac{12+18-3-9}{2} = 8$$

• $f(x,y) = x^2y$

► $D = [0,1] \times [0,3]$

$$\int_0^1 dx \int_0^3 x^2y dy = \int_0^1 dx \left[x^2 \frac{y^2}{2} \right]_0^3 = \int_0^1 \frac{9}{2} x^2 dx = \left[\frac{3}{2} x^3 \right]_0^1 = \frac{3}{2}$$

► $D = [1,2] \times [2,3]$

$$\int_1^2 dx \int_2^3 dy [x^2y] = \int_1^2 dx \left[x^2 \frac{y^2}{2} \right]_2^3 = \int_1^2 \left[\frac{9}{2} x^2 - \frac{4}{2} x^2 \right] dx =$$

$$= \left[\frac{5}{6} x^3 \right]_1^2 = \frac{20}{6} - \frac{5}{6} = \frac{15}{6}$$

• $f(x,y) = \log y$

$$\int_0^1 dx \int_1^2 \log y dy =$$

$$= \int_0^1 dx \left[y \log y - 1 \right]_1^2 = \int_0^1 (2(\log 2 - 1) - 1(\log 1 - 1)) dx$$

$$= 2 \log 2 - 2 + 1 = 2 \log 2 - 1$$

NOTA

$$\int \log y dy = y \log y - \int \frac{1}{y} y dy =$$

$$= y (\log y - 1)$$

• $f(x, y) = \frac{x}{y}$

$D = \{ [0, 1] \times [0, 2] \}$

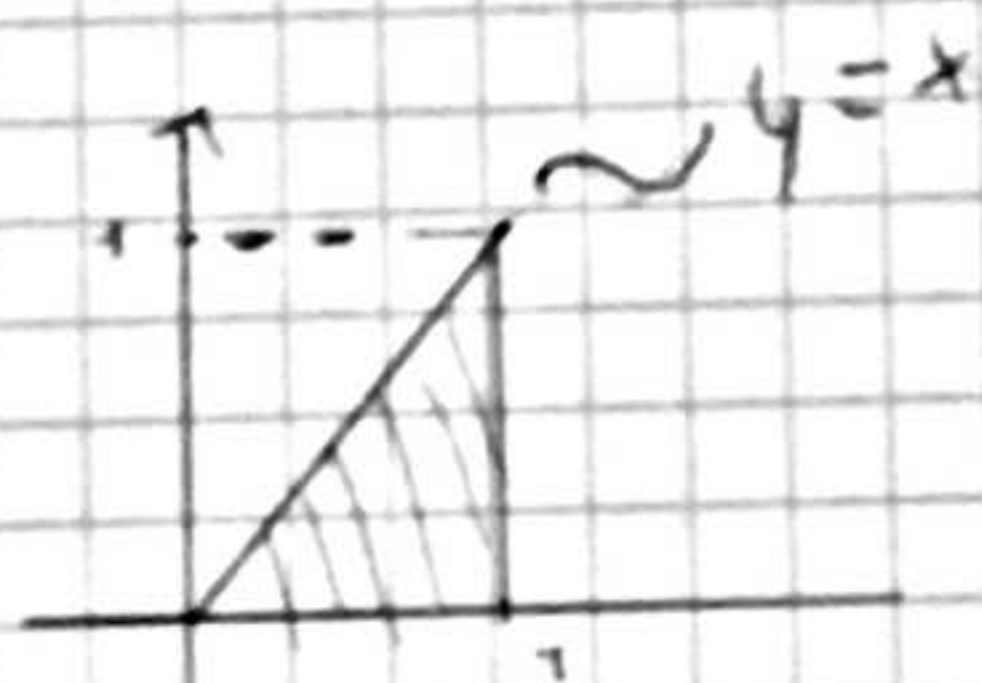
$$\int_0^1 \int_0^2 \frac{x}{y} dx dy = \int_0^1 dy \int_0^2 x \cdot \frac{1}{y} dy = \int_0^1 x \log y \Big|_0^2 dy = \frac{1}{2} \log 2 = \log 2$$

$D = \{ [-2, -1] \times [-2, -1] \}$

$$\int_{-2}^{-1} dy \int_{-2}^{-1} x \cdot \frac{1}{y} dy = \int_{-2}^{-1} x \log y \Big|_{-2}^{-1} dy = \frac{x^2}{2} [\log(-1) - \log(-2)] \Big|_{-2}^{-1}$$

$$= \frac{x^2}{2} \log(1/2) \Big|_{-2}^{-1} = \frac{1}{2} - \frac{4}{2} [\log(1/2)] = \frac{3}{2} \log 2$$

ESERCIZI SU INSIEMI NORMALI

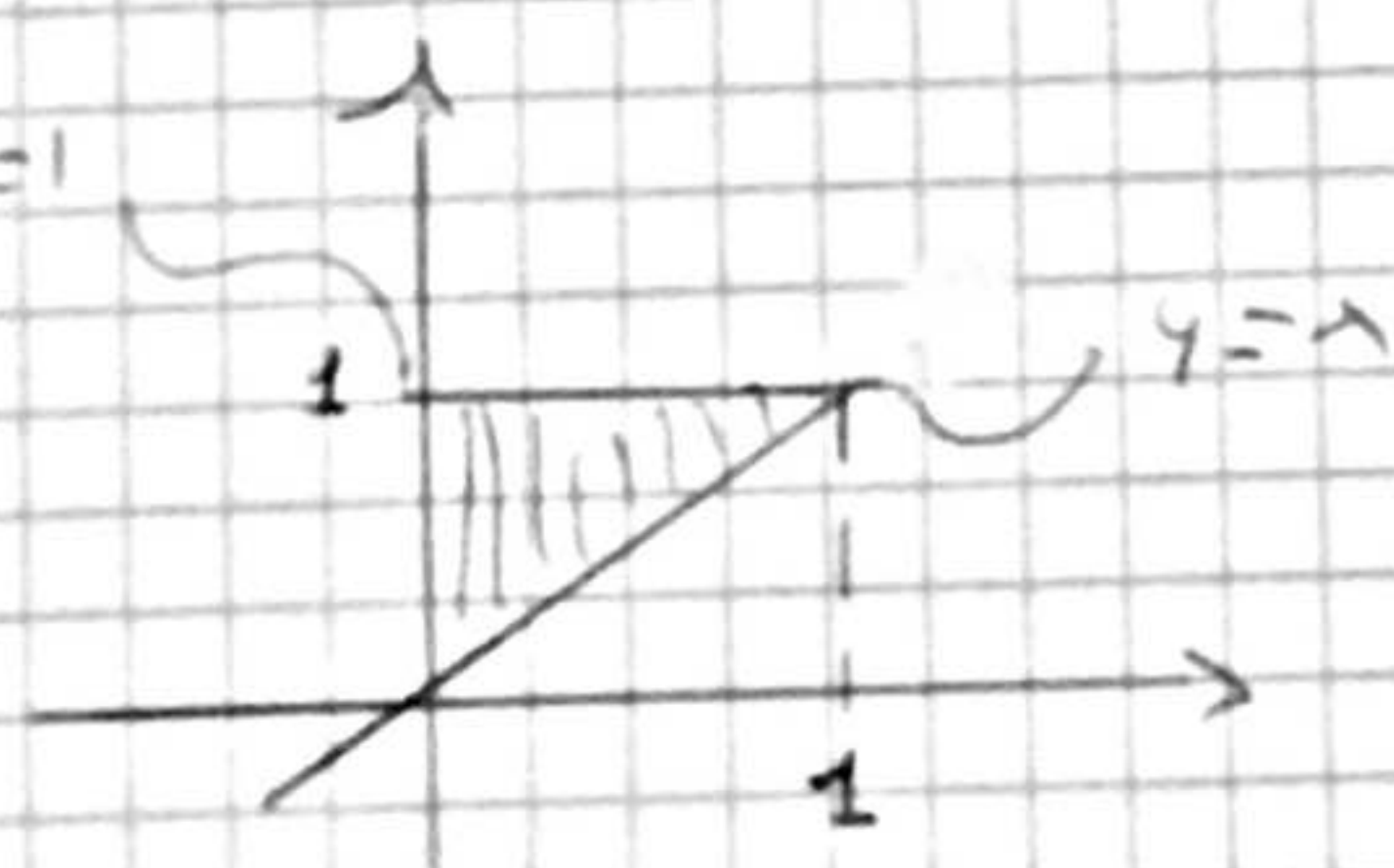


Normale rispetto a x

$D = \{ 0 \leq x \leq 1 ; 0 \leq y \leq x \}$

Normale rispetto a y

$D = \{ 0 \leq y \leq 1 ; y \leq x \leq 1 \}$

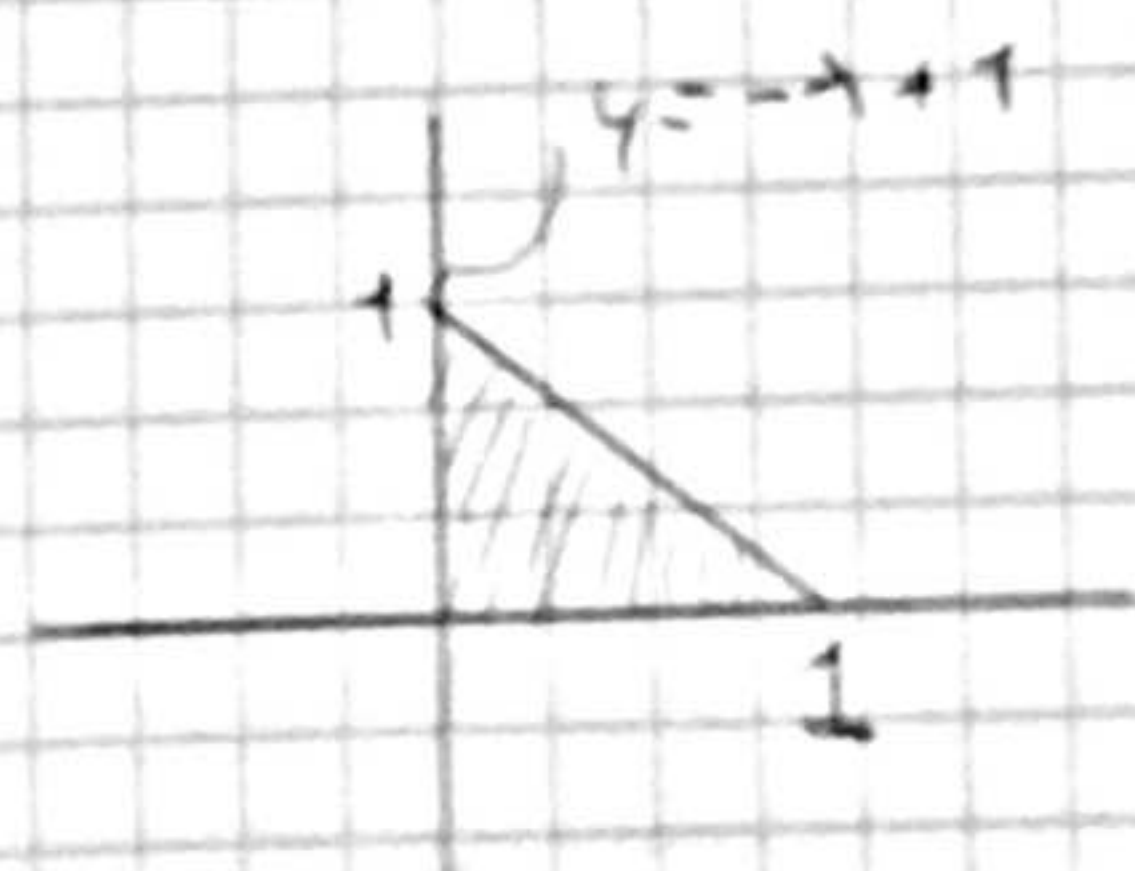


Normale ad x

$D = \{ 0 \leq x \leq 1 ; x \leq y \leq 1 \}$

Normale ad y

$D = \{ 0 \leq y < 1 ; 0 \leq x \leq y \}$

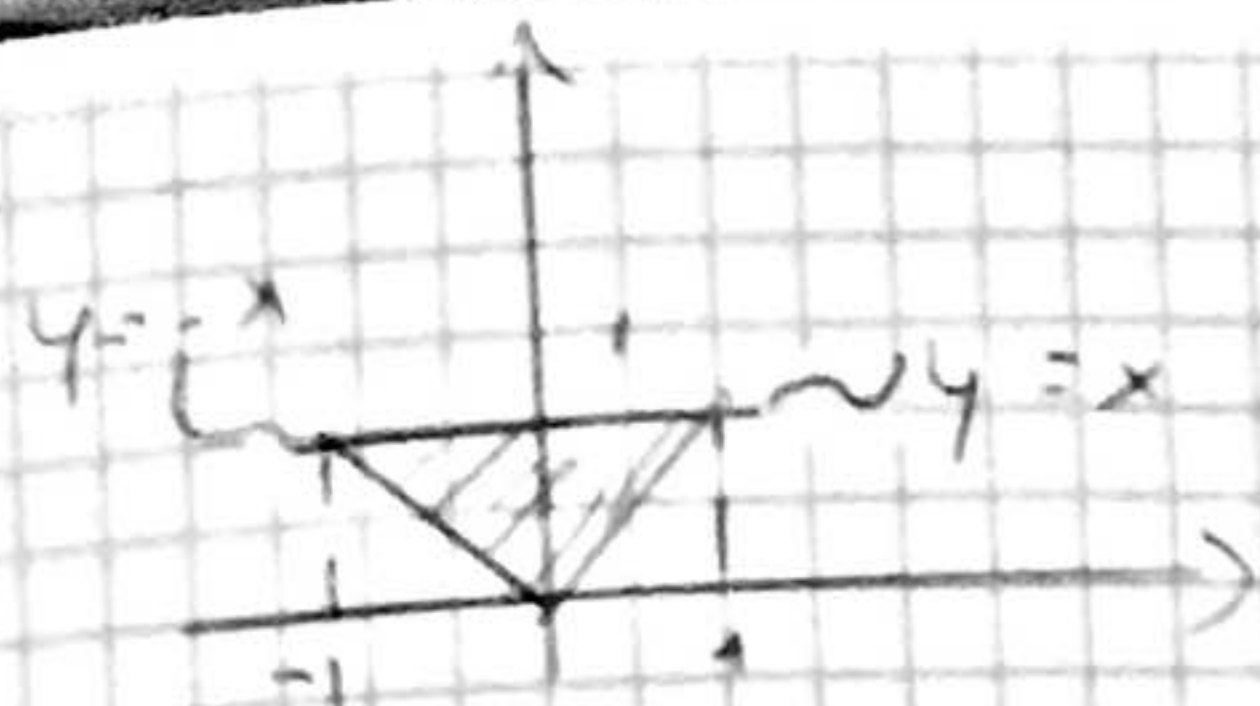


Normale rispetto ad x

$D = \{ 0 \leq x \leq 1 ; 0 \leq y \leq 1-x \}$

Normale rispetto ad y

$D = \{ 0 \leq y \leq 1 ; 0 \leq x \leq 1-y \}$



Insideri normali ad x

$$D = \{-1 \leq x \leq 1; -x \leq y \leq x\}$$

Insideri normali ad y

$$D = \{0 \leq y \leq 1; -y \leq x \leq y\}$$

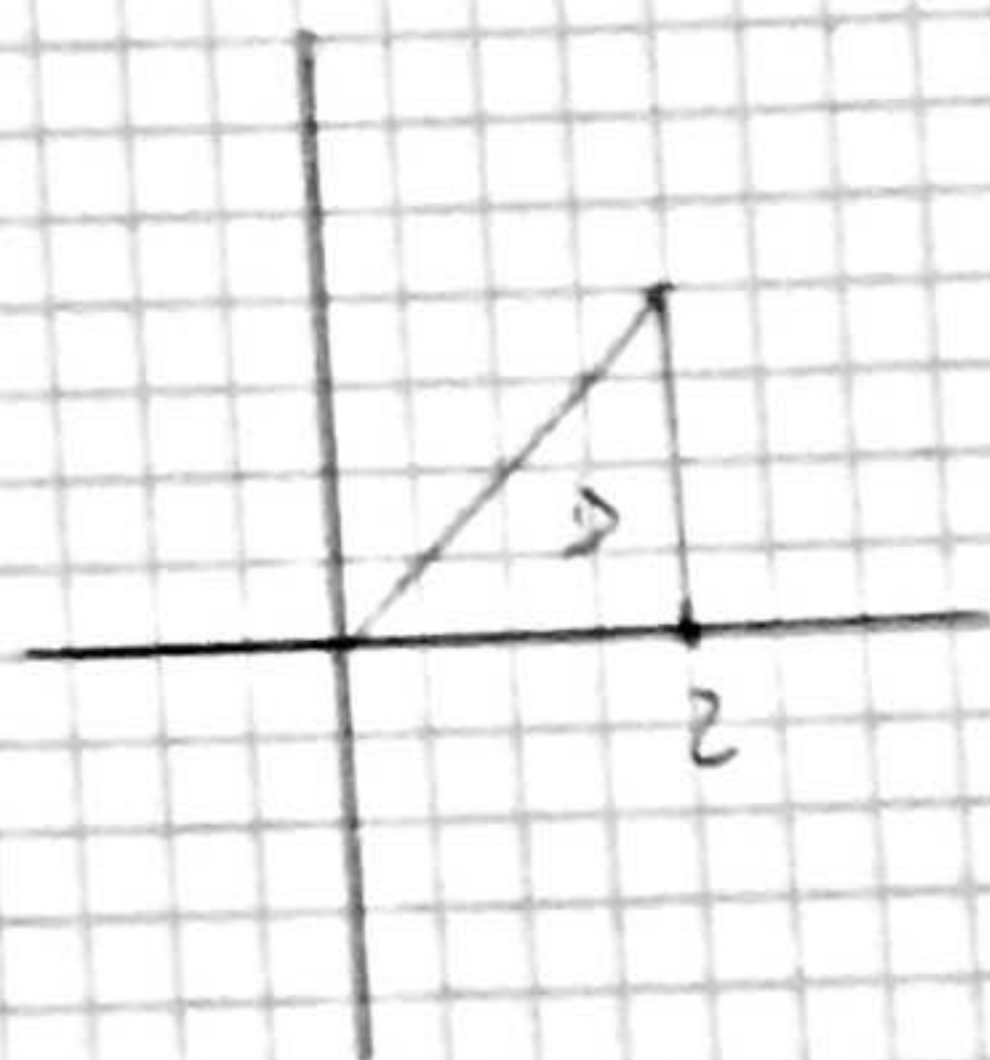
INTEGRALI DOPPI ESERCIZI

$$D = \{x \in [0, 2] \mid 0 \leq y \leq x\}$$

$$\rightarrow f(x, y) = x + 6y$$

$$\int_0^2 dx \int_0^x (x + 6y) dy = \int_0^2 \left[xy + \frac{6}{2} y^2 \right]_0^x dx =$$

$$= \int_0^2 dx (x^2 + 3x^2) = \left[\frac{4}{3} x^3 \right]_0^2 = \frac{32}{3}$$



$$\rightarrow f(x, y) = x^3$$

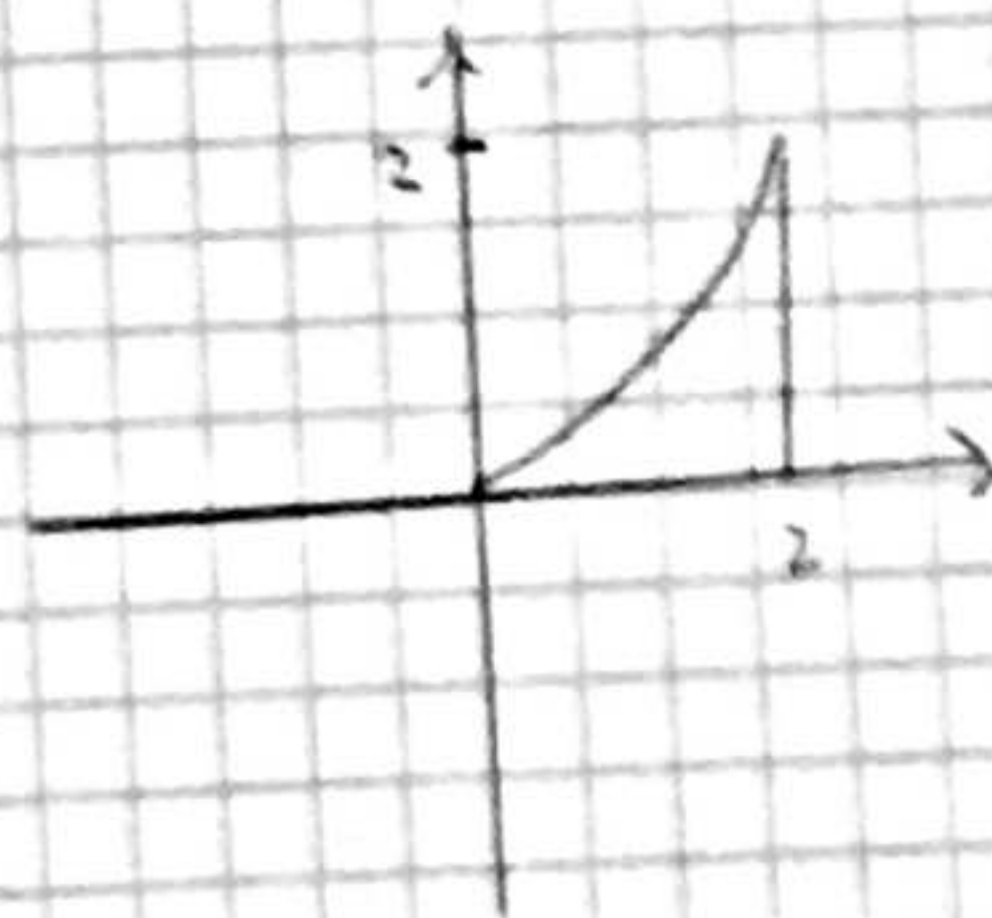
$$\int_0^2 dx \int_0^x x^3 dy = \int_0^2 x^3 y \Big|_0^x dx = \left[\frac{x^5}{5} \right]_0^2 = \frac{32}{5}$$

$$D = \{x \in [0, 2] \mid 0 \leq y \leq x^2\}$$

$$f(x, y) = xy$$

$$\int_0^2 dx \int_0^{x^2} xy dy = \int_0^2 \left[\frac{xy^2}{2} \right]_0^{x^2} dx =$$

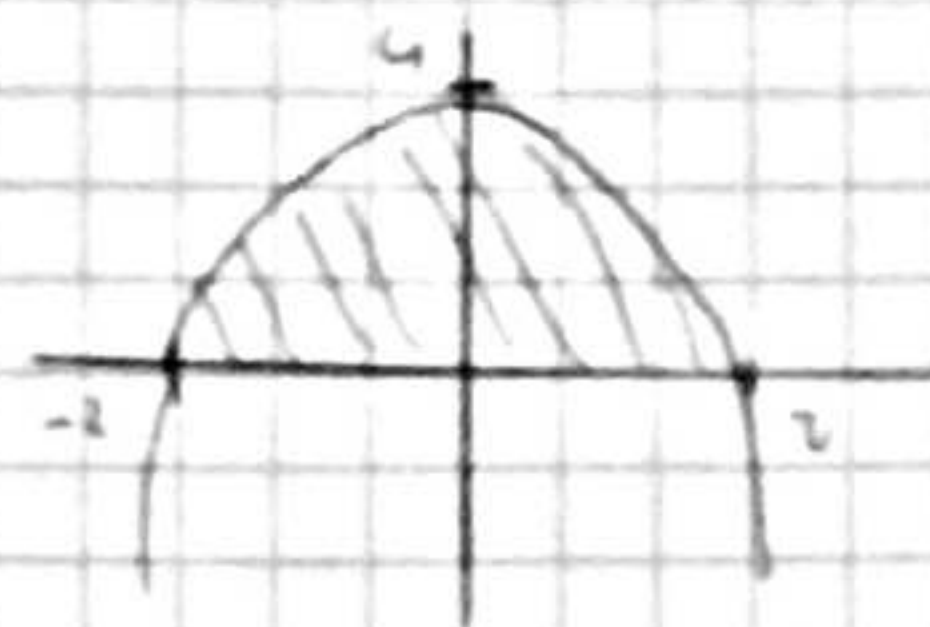
$$= \int_0^2 \frac{x^5}{2} = \left[\frac{x^6}{12} \right]_0^2 = \frac{64}{12} = \frac{16}{3}$$



$$\int_0^2 dx \int_0^{x^2} x^3 dy = \int_0^2 x^3 y \Big|_0^{x^2} dx = \int_0^2 x^5 dx = \left[\frac{x^6}{6} \right]_0^2 = \frac{32}{3}$$

• Sei $D = \{0 \leq y \leq 4 - x^2\}$

$f(x, y) = x^2 y \quad \hookrightarrow -2 \leq x \leq 2$



$$\int_{-2}^2 dx \int_0^{4-x^2} x^2 y dy = \int_{-2}^2 x^2 \left[\frac{y^2}{2} \right]_0^{4-x^2} dx = \int_{-2}^2 x^2 \frac{(4-x^2)^2}{2} dx =$$

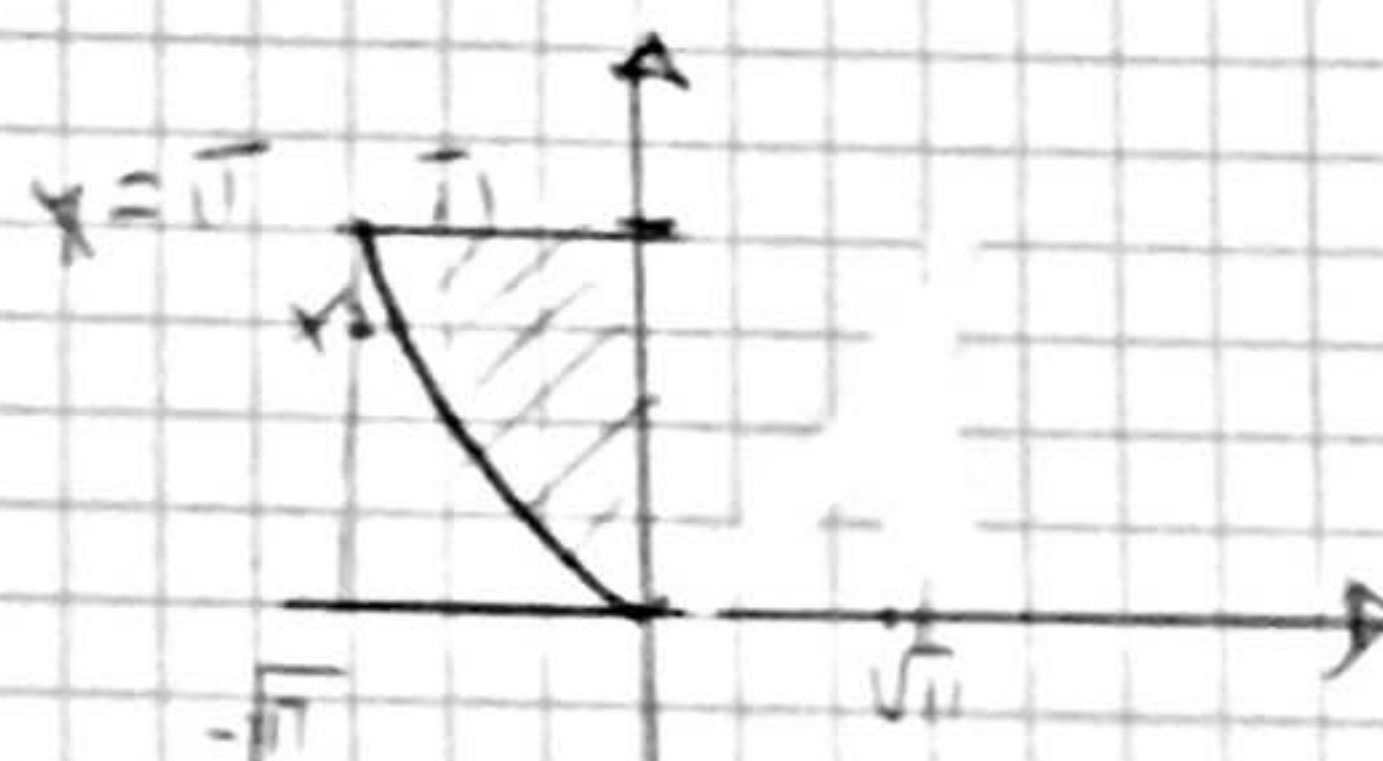
funktion \rightarrow $= \int_0^2 x^2 (4-x^2)^2 dx = \int_0^2 16x^2 dx + \int_0^2 x^6 dx - \int_0^2 8x^4 dx =$

$$= \left[\frac{16}{3} x^3 \right]_0^2 + \left[\frac{1}{7} x^7 \right]_0^2 - \left[\frac{8}{5} x^5 \right]_0^2 = \frac{16}{3} \cdot 8 + \frac{1}{7} \cdot 2^7 - \frac{8}{5} \cdot 2^5 = \frac{128}{3} + \frac{128}{7} - \frac{256}{5} = \frac{2}{105} = \frac{1024}{105}$$

• $D = \{x \in [-\sqrt{\pi}, 0], x^2 + y \leq \pi\}$

$$\int_{-\sqrt{\pi}}^0 dx \int_{x^2}^{\pi} x \cos y dy = \int_{-\sqrt{\pi}}^0 -x \sin y \Big|_{x^2}^{\pi} dx =$$

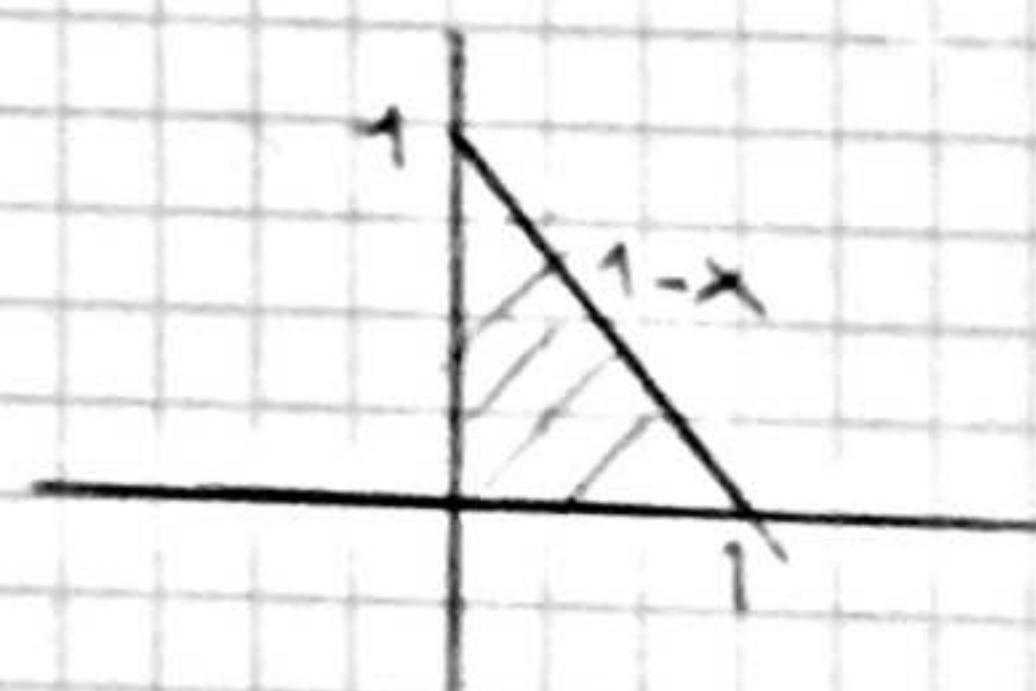
$$= \int_{-\sqrt{\pi}}^0 -x \sin x^2 dx = -\frac{1}{2} \int_{\pi}^0 \sin u du = \frac{\cos u}{2} \Big|_{\pi}^0 = \frac{\cos x^2}{2} \Big|_{-\sqrt{\pi}}^0 = \frac{1}{2} - \frac{1}{2} = 0$$



$f(x, y) = x - 2y$

$D = \{x \geq 0, y \geq 0; x + y \leq 1\}$

$0 \leq x \leq 1; 0 \leq y \leq 1-x$



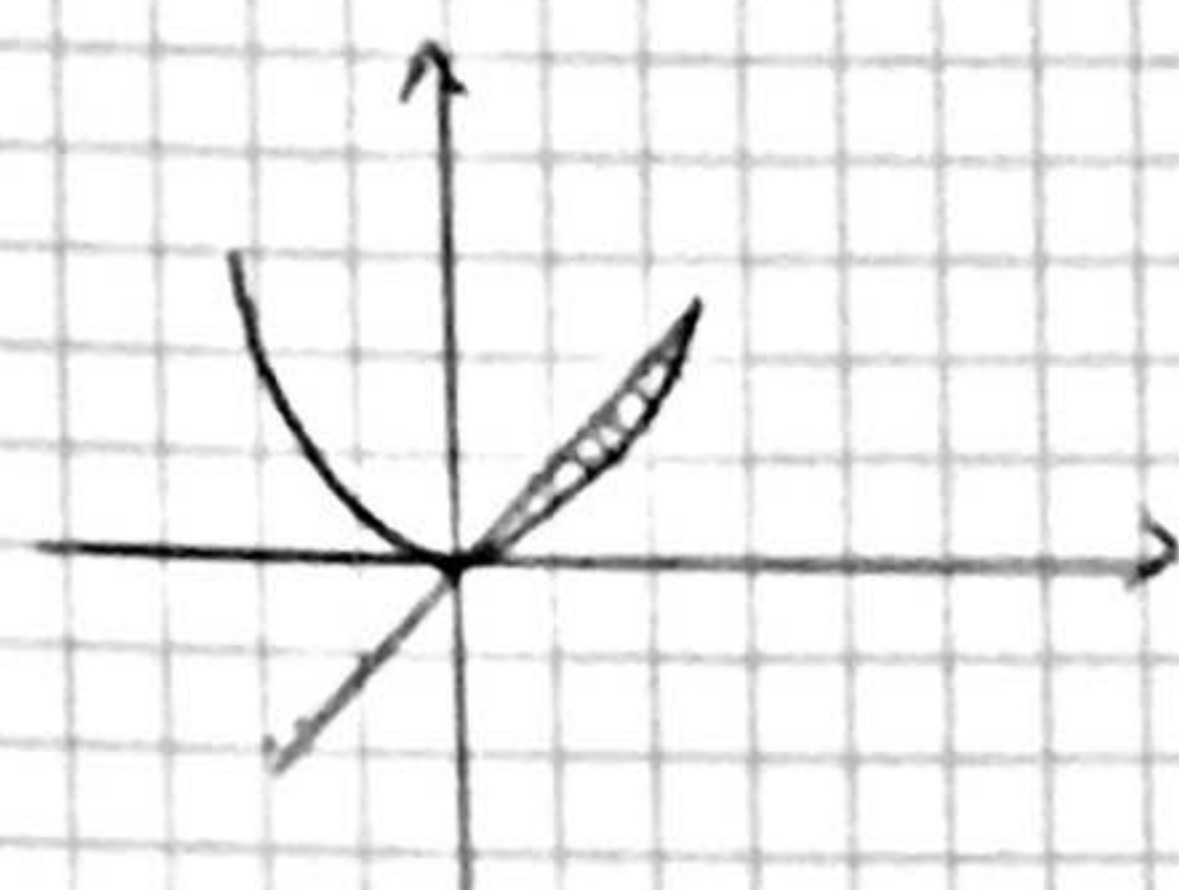
$$\int_0^1 dx \int_0^{1-x} (x - 2y) dy = \int_0^1 \left[xy - y^2 \right]_0^{1-x} dx = \int_0^1 (x(1-x) - (1-x)^2) dx =$$

$$= \int_0^1 (x - x^2 - 1 + 2x) dx = \int_0^1 (-2x^2 + 3x - 1) dx =$$

$$= \left[-\frac{2}{3} x^3 + \frac{3}{2} x^2 - x \right]_0^1 = -\frac{2}{3} + \frac{3}{2} - 1 = \frac{-4 + 9 - 6}{6} = -\frac{1}{6}$$

$$f(x, y) = x$$

$$D = \{x^2 \leq y \leq x\}$$



$$\int_0^1 dx \int_{x^2}^x (x^2 + xy^2) dy =$$

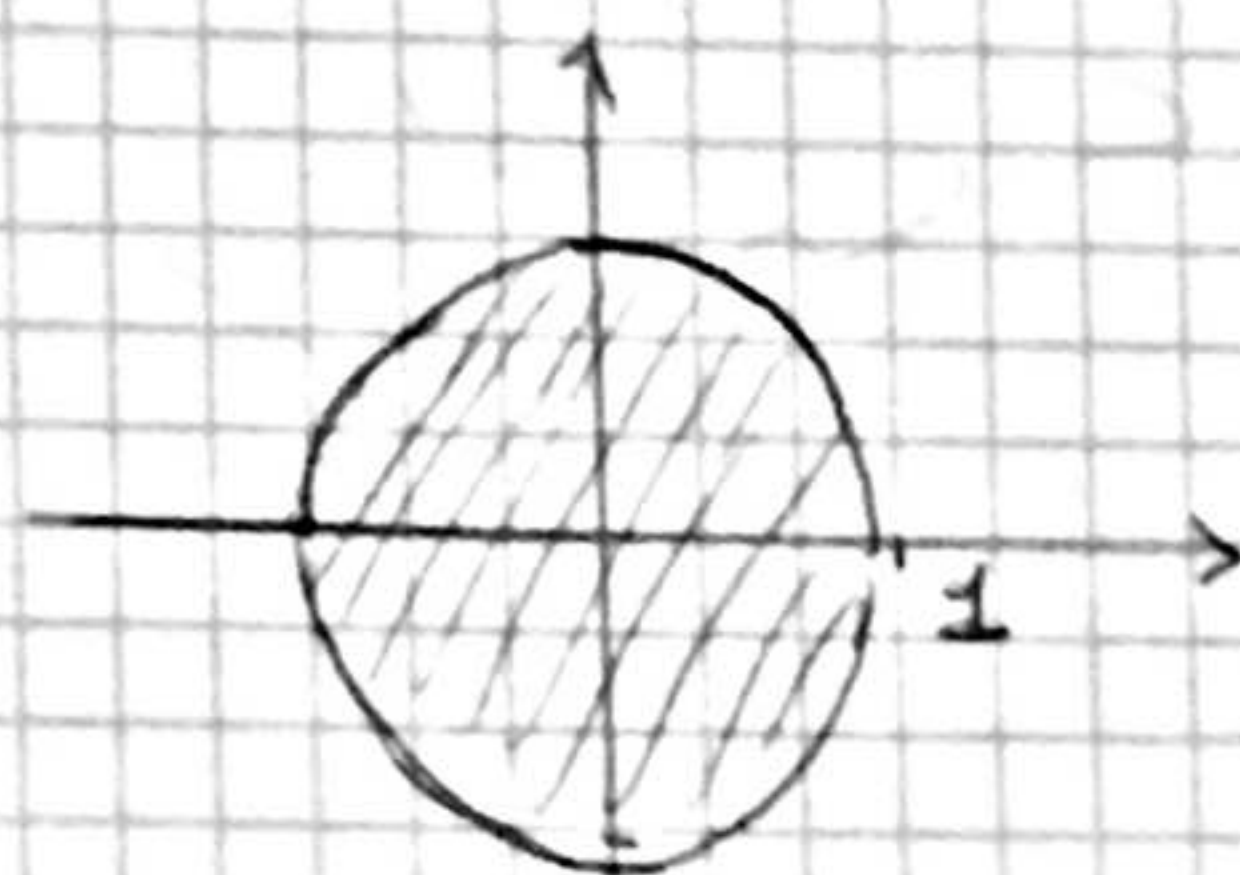
$$= \int_0^1 dx \left[x^2 y + x \frac{y^3}{3} \right]_{x^2}^x = \int_0^1 \left(x^3 - x^4 + \frac{x^4}{3} - \frac{x^7}{3} \right) dx =$$

$$= \frac{x^4}{4} - \frac{x^5}{5} + \frac{x^5}{15} - \frac{x^8}{24} \Big|_0^1 = \frac{20 - 24 + 8 - 5}{120} = \frac{3}{120} = \frac{1}{40}$$

$$f(x, y) = x$$

$$D = \{x^2 + y^2 \leq 1\}$$

$$\hookrightarrow -1 \leq x \leq 1 \quad -1 \leq y \leq 1$$



$$\int_{-1}^1 dx \int_{-1}^1 x dy = \int_{-1}^1 (x + x) dx = \frac{x^2}{2} \Big|_{-1}^1 = \frac{1}{2} - \frac{1}{2} = 0$$

- Sia ora $f(x, y) = y \rightarrow \int_{-1}^1 dx \int_{-1}^1 y dy = \int_{-1}^1 \frac{y^2}{2} \Big|_{-1}^1 dx = 0$

- Sia ora $f(x, y) = |x|$

$$\int_0^1 dx \int_0^{\sqrt{1-x^2}} x dy = \int_0^1 x y \Big|_0^{\sqrt{1-x^2}} dx = 2 \int_0^1 x \sqrt{1-x^2} = -\frac{1}{3} (1-x^2)^{3/2} \Big|_0^1 = \frac{2}{3}$$

$$= \int_{-1}^0 dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x dy = -2 \int_0^1 x \sqrt{1-x^2} = \frac{2}{3} (1-x^2)^{3/2} \Big|_0^{-1} = \frac{2}{3}$$

Somma le due parti e l'integrale è $\frac{4}{3}$

Via più semplice con le coordinate polari

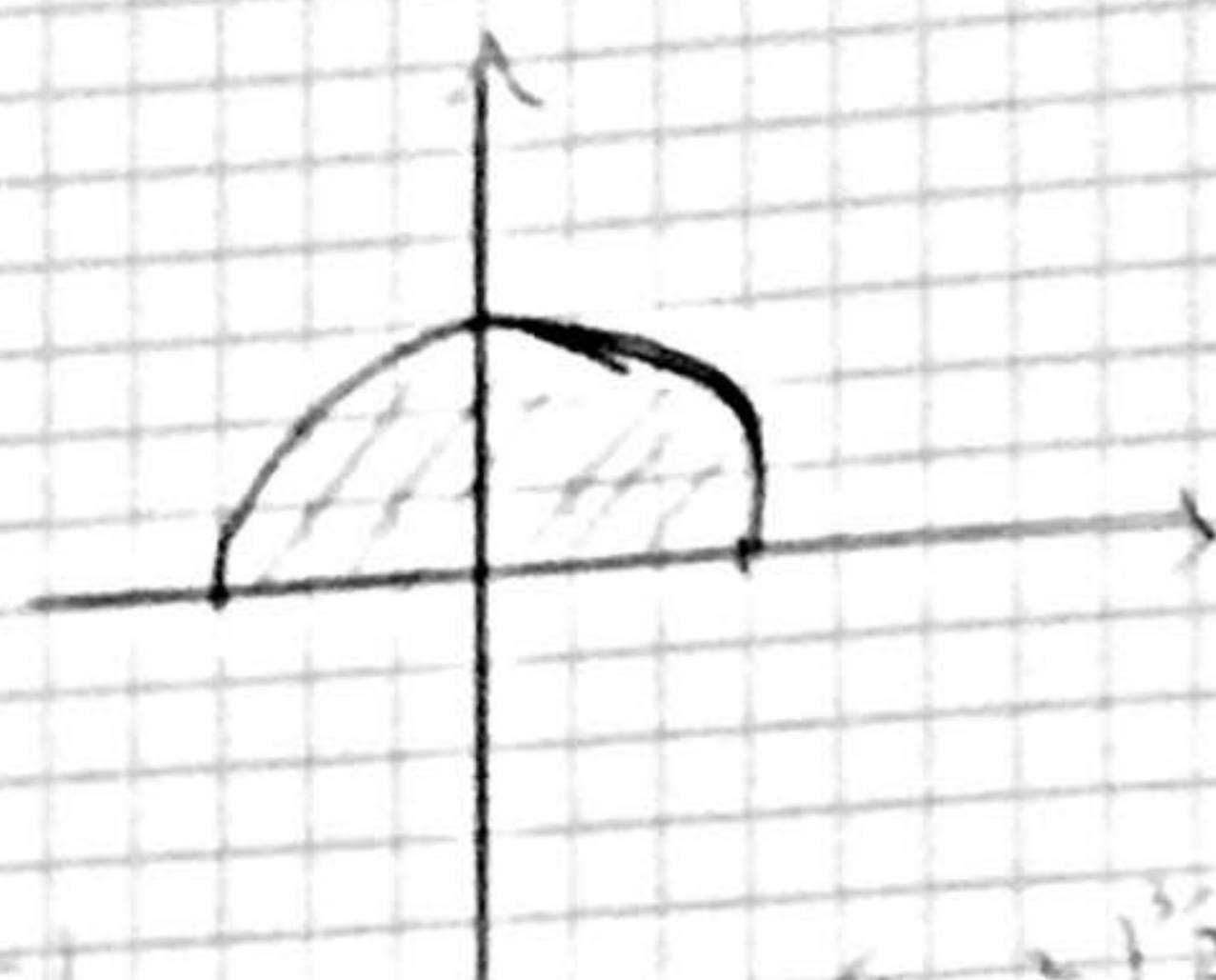
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$$f(x, y) = x$$

lo scrivo come normale ad x

$$D = \{ -1 \leq x \leq 1; 0 \leq y \leq \sqrt{1-x^2} \}$$

$$\int_{-1}^1 dx \int_0^{\sqrt{1-x^2}} x dy = \int_{-1}^1 x y \Big|_0^{\sqrt{1-x^2}} dx = \int_{-1}^1 x \sqrt{1-x^2} = \frac{1}{3} (1-x^2)^{3/2} \Big|_{-1}^1 = 0$$



$$f(x, y) = y$$

$$\int_{-1}^1 dx \int_0^{\sqrt{1-x^2}} y dy = \int_{-1}^1 \frac{y^2}{2} \Big|_0^{\sqrt{1-x^2}} dx = \left[\frac{x}{2} - \frac{x^3}{6} \right]_{-1}^1 = 1 - \frac{2}{6} = \frac{2}{3}$$

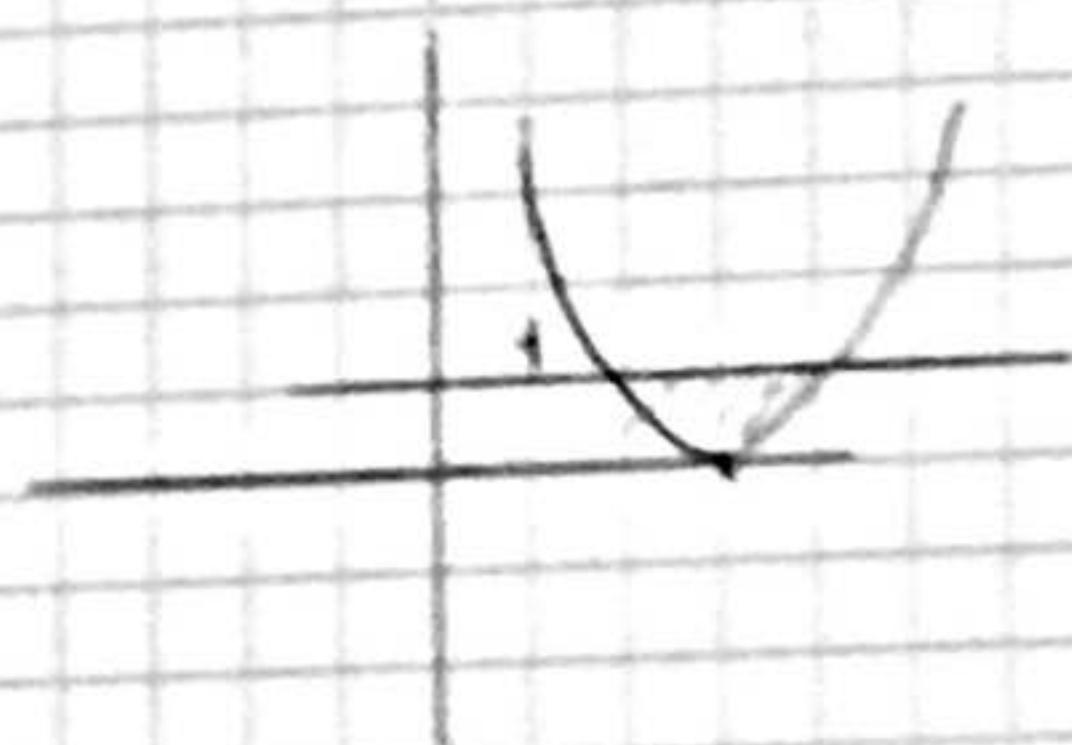
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$$f(x, y) = y^6 \quad D = \{ x^2 - 6x + 9 \leq y \leq 1 \}$$

$$\begin{cases} y = 1 \\ y = x^2 - 6x + 9 \\ 1 = x^2 - 6x + 9 \\ 0 = x^2 - 6x + 8 \end{cases}$$

$$\Delta = 36 - 32 = 4$$

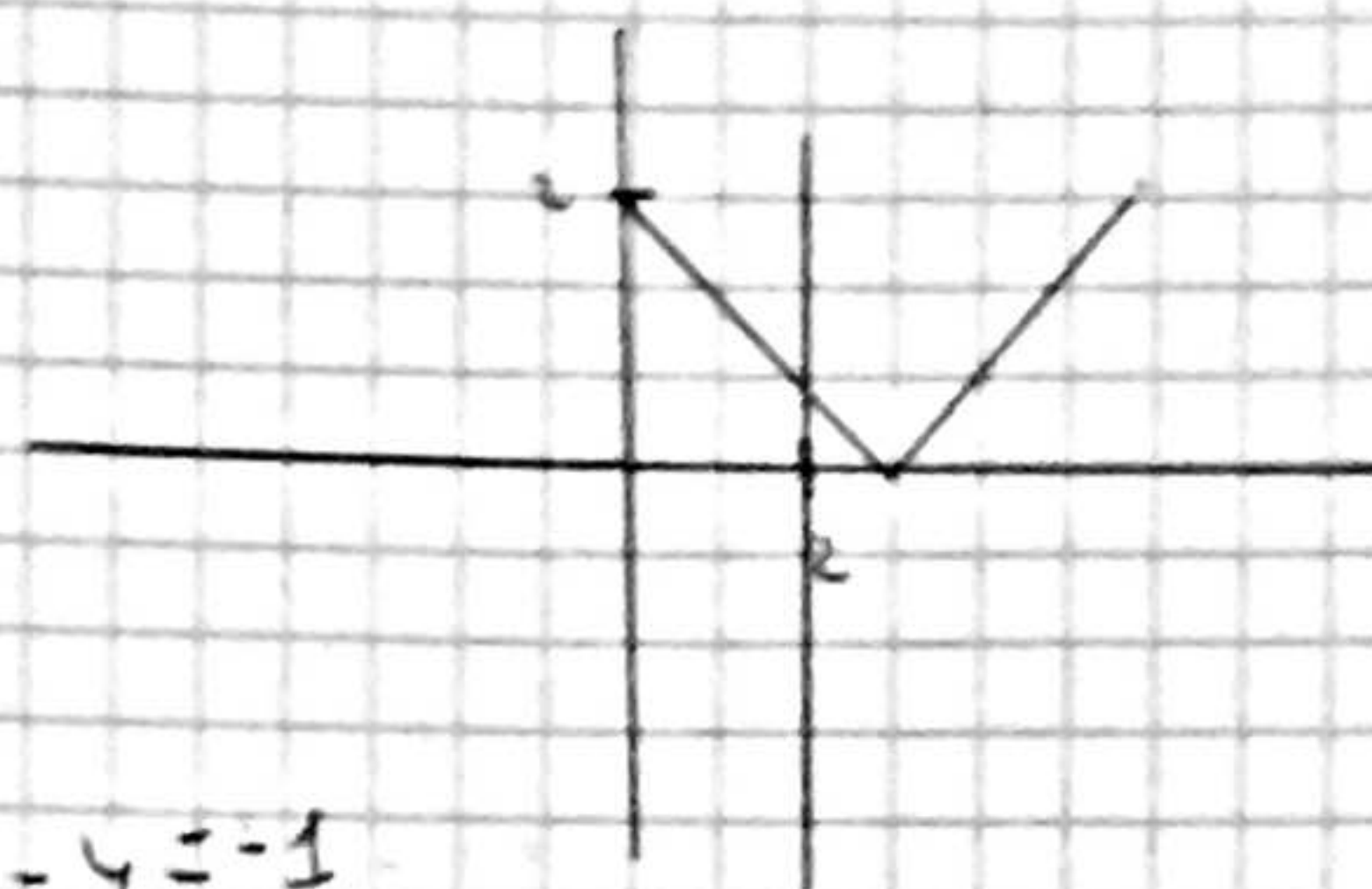
$$x_{1,2} = \frac{6 \pm \sqrt{4}}{2} = \frac{6 \pm 2}{2}$$



$$\begin{aligned} \int_2^4 dx \int_{(x-3)^2}^1 y^6 dy &= \int_2^4 \left(\frac{1}{7} - \frac{(x-3)^{14}}{15 \cdot 2} \right) dx = \left[\frac{x}{7} - \frac{(x-3)^{15}}{15 \cdot 2} \right]_2^4 = \\ &= \frac{4}{7} - \frac{1}{15 \cdot 2} - \left(\frac{2}{7} - \frac{1}{15 \cdot 2} \right) = \frac{2}{7} - \frac{2}{15 \cdot 2} = \frac{30-2}{15 \cdot 7} = \frac{28}{105} = \frac{4}{15} \end{aligned}$$

$$f(x, y) = y - 3$$

$$D = \{ |y-3| \leq x \leq 2 \}$$

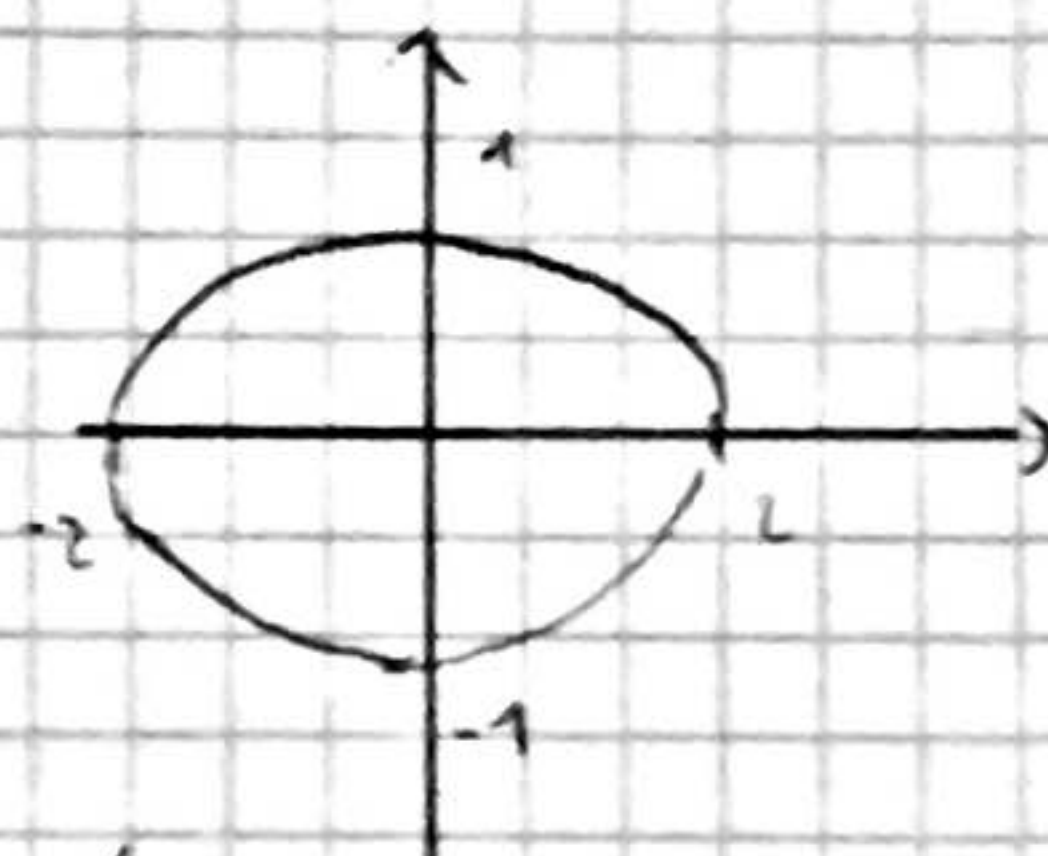


$$\begin{cases} x=2 \\ |y-3|=x \end{cases} \begin{cases} x = -y+3 \rightarrow -y = -1 \\ x = y-3 \rightarrow y = 5 \end{cases}$$

$$\int_0^2 dx \int_1^5 y-3 dy = \int_0^2 \left[\frac{y^2}{2} - 3y \right]_1^5 dx = \int_0^2 \left(\frac{25}{2} - 15 - \frac{1}{2} + 3 \right) dx = 0$$

$$f(x, y) = 1$$

$$D = \{ x^2 + 4y^2 \leq 4 \}$$



$$\begin{aligned} x &= u \\ y &= \frac{v}{2} \end{aligned} \quad J = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \rightarrow \det J = \frac{1}{2}$$

$$\int_{u^2+v^2 \leq 4} \frac{1}{2} du dv = \frac{4\pi}{2} = 2\pi$$

SWIVUPARE FIMO A22'ORDINE 4

$$\begin{aligned} (1+x^2+y^2+xy)^{1/4} - 1 &= \frac{1}{4} (x^2+y^2+xy) + \frac{1}{4} \left(\frac{1}{4} - 1 \right) \left[x^4+y^4+2x^3y^2y^3 \right. \\ &\quad \left. + x^2y^2+2x^2y^2+0(x^2+y^2+xy)^2 \right] - \frac{1}{2} = \\ &= \frac{1}{4} (x^2+y^2+xy) + \left(-\frac{3}{32} [x^2+y^2+xy]^2 \right) + O(x^2+y^2) \end{aligned}$$

$$f_x = \frac{2x+y}{4(x^2+xy+y^2+1)^{3/4}} \rightarrow f_{xx} = \frac{1}{2(x^2+xy+1)^{3/4}} - \frac{3(2x+y)^2}{16(x^2+xy+y^2+1)^{7/4}}$$

$$f_y = \frac{x+2y}{4(x^2+xy+y^2+1)^{3/4}} \rightarrow f_{yy} = \frac{1}{2(x^2+xy+y^2+1)^{3/4}} - \frac{3(x+2y)^2}{16(x^2+xy+y^2+1)^{7/4}}$$

$$f_{xy} = \frac{1}{4(x^2+xy+y^2+1)^{3/4}} - \frac{3(2x+y)(x+2y)}{16(x^2+xy+y^2+1)^{7/4}}$$

$$H_f \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} \Rightarrow H_f(0,0) \begin{pmatrix} 1/2 & 1/4 \\ 1/4 & 1/2 \end{pmatrix} \Rightarrow \det H_f(0,0) = \frac{1}{4} - \frac{1}{16} = \frac{3}{16}$$

λ de limite positiva $\Rightarrow (0,0)$ e min local

Fare lo sviluppo fino all'ordine 4^o di

$$\sin(x-y) \arctan(x-y)$$

$$\text{sic } (x-y) = z$$

Allora procedo con lo sviluppo di

$$\sin z \arctan(z) = \left[z - \frac{z^3}{6} + o(z^5) \right] \left[z - \frac{z^3}{3} + o(z^5) \right]$$

$$\begin{aligned} \rightarrow \sin(x-y) \arctan(x-y) &= (x-y)^2 - \frac{(x-y)^4}{3} - \frac{(x-y)^4}{6} + o((x-y)^4) \\ &= (x-y)^2 - \frac{(x-y)^4}{3} - \frac{(x-y)^4}{6} + o((x^2+y^2)^2) = \\ &= (x-y)^2 - \frac{1}{2} (x-y)^4 + o((x^2+y^2)^2) \end{aligned}$$

$$f_x = 2 \cos(x-y) \arctan(x-y) + \sin(x-y) \frac{1}{(x-y)^2+1}$$

$$f_{xx} = -2(x-y) \sin(x-y) + \frac{2 \cos(x-y)}{(x-y)^2+1} - \sin(x-y) \tan''(x-y)$$

$$f_{xy} = \frac{2(x-y) \sin(x-y)}{((x-y)^2+1)^2} - \frac{2 \cos(x-y)}{(x-y)^2+1} + \sin(x-y) \arctan(x-y)$$

$$f_y = -\cos(x-y) \arctan(x-y) - \frac{\sin(x-y)}{(x-y)^2+1}$$

$$f_{yy} = -\frac{2(x-y) \sin(x-y)}{((x-y)^2+1)^2} + \frac{2 \cos(x-y)}{(x-y)^2+1} - \sin(x-y) \arctan(x-y)$$

$$H_f(0,0) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

$$\det H_f = 2+2=4 \\ \text{tr} = 4$$

la matrice è definita positiva allora ho min locale

• $\arctan(xy^3 + \sin y) \Rightarrow$ fare lo sviluppo fino all'ordine 4

fanga $z = xy^3 + \sin y$

$$\arctan z = z - \frac{z^3}{3} + O(z^5)$$

$$\begin{aligned} \rightarrow \arctan(xy^3 + \sin y) &= xy^3 + \sin y - \frac{(xy^3 + \sin y)^3}{3} + O((xy^3 + \sin y)^5) \\ &= xy^3 + \sin y - \frac{\sin^3 y}{3} + O((xy^3 + \sin y)^3) \\ &= xy^3 + y - \frac{y^3}{6} - \frac{y^3}{3} + O((xy^3 + y)^3) \\ &= xy^3 + y - \frac{y^3}{2} + O((x^2 y^2)^2) \end{aligned}$$

Perché c'è il termine al primo ordine dello sviluppo allora il gradiente non è nullo quando calcolato nell'origine e quindi non è stazionario.

ESERCIZI INTEGRALI TRIPLI

• $D = \{ [0, 1] \times [0, 2] \times [0, 3] \}$

$f(x, y, z) = x \rightarrow \int_0^1 dx \int_0^2 dy \int_0^3 x \, dz = \int_0^1 dx \int_0^2 3x \, dy = \int_0^1 6x \, dx = 3$

• $D = \{ [0, \pi] \times [0, \pi] \times [0, \pi] \}$

$f(x, y, z) = \sin y \rightarrow \int_0^\pi dx \int_0^\pi dy \int_0^\pi \sin y \, dz = \int_0^\pi dx \int_0^\pi \pi \sin y \, dy = -\int_0^\pi \pi \cos y \Big|_0^\pi dx = \pi \int_0^\pi dx = 2\pi^2$

• $D = \{ (x, y) \in [0, 1] \times [0, 1], 0 \leq z \leq xy \}$

$f(x, y, z) = y$

$$\begin{aligned} \int_0^1 dx \int_0^1 dy \int_0^{xy} y \, dz &= \int_0^1 dx \int_0^1 y^2 x \, dy = \\ &= \int_0^1 \left[\frac{y^3}{3} x \right]_0^1 dx = \int_0^1 \frac{x}{3} dx = \frac{x^2}{6} \Big|_0^1 = \frac{1}{6} \end{aligned}$$

$$\bullet D = \{x^2 + y^2 \leq 4, z \in [-1, 2]\}$$

$$f(x, y, z) = x^2$$

$$\det J = \begin{pmatrix} \cos \theta & \sin \theta \\ -\rho \sin \theta & \rho \cos \theta \end{pmatrix}$$

uso la polar

$$\int_{x^2+y^2 \leq 4} dx dy \int_{-1}^2 x^2 dz = \int_{x^2+y^2 \leq 4} \left. x^2 z \right|_{-1}^2 dx dy = \int_{x^2+y^2 \leq 4} 3x^2 dx dy =$$

$$= 3 \int_0^2 \rho d\rho \int_0^{2\pi} \rho \cdot \rho^2 \cos^2 \theta d\theta = 3\pi \int_0^2 \rho^3 d\rho = 3\pi \left. \frac{\rho^4}{4} \right|_0^2 = 12\pi$$

$$\bullet D = \{x^2 + y^2 \leq 4, z \in [-1, 2]\}$$

$$f(x, y, z) = x^2 + y^2 + z^2$$

$$\int_{x^2+y^2 \leq 4} dx dy \int_{-1}^2 x^2 + y^2 + z^2 dz = \int_{x^2+y^2 \leq 4} \left[x^2 z + y^2 z + \frac{z^3}{3} \right]_{-1}^2 dx dy =$$

$$= \int_{x^2+y^2 \leq 4} (3x^2 + 3y^2 + 1) dx dy =$$

sprezzo nella somma
di 2 integrali

$$= 3 \int_0^{2\pi} \int_0^2 \rho \cdot \rho^2 d\rho d\theta + \int_{x^2+y^2 \leq 4} 3 dx dy = 24\pi + 12\pi = 36$$

viene dalla coordinate
polari dove

$$\det J = \begin{pmatrix} \cos \theta & \sin \theta \\ -\rho \sin \theta & \rho \cos \theta \end{pmatrix}$$

area archio
di raggio 2

$$D = \{ x^2 + y^2 + z^2 \leq 4 \}$$

\hookrightarrow sfera di raggio 2 \Rightarrow lo scriviamo come normale rispetto a z

$$f(x, y, z) = z - 1$$

$$\bar{D} = \{ -2 \leq z \leq 2; x^2 + y^2 \leq 4 - z^2 \}$$

$$\int_{x^2 + y^2 + z^2 \leq 4} (z - 1) dx dy dz = \int_{-2}^2 dz \int_{x^2 + y^2 \leq 4 - z^2} (z - 1) dx dy$$

\uparrow
area cerchio raggio $\sqrt{4 - z^2}$

$$= \pi \int_{-2}^2 dz (4 - z^2)(z - 1)$$

$$= \pi \int_{-2}^2 dz [4z - 4 - z^3 + z^2] =$$

$$= \pi \left[\cancel{4} \frac{z^2}{2} - 4z - \frac{z^4}{4} + \frac{z^3}{3} \right]_{-2}^2 = \pi \left[8 - 8 - \cancel{4} + \frac{8}{3} - \left(8 - 8 - \cancel{4} - \frac{8}{3} \right) \right]$$

$$= \pi \left[\frac{16}{3} - 16 \right] = -\frac{32\pi}{3}$$