Exercises in stochastic analysis

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In linea di massima (con possibili eccezioni o dimenticanze):
- gli esercizi marcati con “svolto” sono stati svolti nelle esercitazioni precedenti;
- gli esercizi marcati con “possibile” potrebbero essere svolti nell’esercitazione successiva;
- gli esercizi marcati con “consigliato” sono stati suggeriti a lezione, ma non svolti.

Si consiglia di controllare l’ultima versione (almeno per gli esercizi marcati come svolti), poiché vi possono essere alcune correzioni agli enunciati.

1 Laws and marginals

Exercise 1 (Consigliato). Let τ be a random variable (r.v.) with values in $[0,+\infty[$, define the (continuous-time) waiting process X as $X_t = 1_{\tau<t}$. Compute the m-dimensional distributions of X, for any m integer. Try to figure out the law of X as a r.v. with values on the space $\{0,1\}^{[0,+\infty[}$.

Exercise 2 (Svolto per la prima parte). Exhibit two processes, which are not equivalent but have the same 1- and 2-dimensional distributions. Exhibit a family of compatible 2-dimensional distributions, which does not admit a process having those distributions as 2-dimensional marginals.

Exercise 3. Let $(F,\mathcal{F})$ be a measurable space, G a subset (not necessarily measurable) of F, define the restriction of $\mathcal{F}$ to G as $\mathcal{F}_{|G} := \{A\cap G | A\in\mathcal{F}\}$; it is easy to see that $\mathcal{F}_{|G}$ is a σ-algebra on G.
Now consider $(E,\mathcal{E})$ a measurable space (morally, the state space for some continuous process), call $C = C([0,T];E)$. Show that $\mathcal{B}(C) = (\mathcal{E}^{[0,T]})_{|C}$. This allows to define the law of a continuous process on C, directly from the definition of law for a generic process.

2 Trajectories and filtrations

Exercise 4. Let X be a continuous process. Prove that its natural filtration $(\mathcal{F}_t)_t$ ($\mathcal{F}_t = \sigma(X_s | s\leq t)$) is left-continuous, that is $\mathcal{F}_t = \bigvee_{s<t}\mathcal{F}_s$.

Exercise 5 (Possibile). Let X, Y be two processes, a.e. continuous and modifications one of each other. Show that they are indistinguishable.
Let X be a process, such that a.e. trajectory is continuous. Show that there exists a

process U, indistinguishable of X, which is everywhere continuous, i.e. every trajectory is continuous.

Exercise 6. Let X be a progressively measurable bounded process. Show that the process $(Y_t = \int_0^t X_s ds)_t$ is progressively measurable (with respect to the same filtration).

Exercise 7. Let X, Y be two processes, modifications one of each other. Suppose that X is adapted to a filtration $\mathcal{F} = (\mathcal{F}_t)_t$ and that $\mathcal{F}_0$ is completed, that is, contains all the P-negligible sets. Prove that also Y is adapted to $\mathcal{F}$.

Exercise 8. Let X be a process, adapted to a filtration $\mathcal{F}$. Fix $t \geq 0$ and $(t_n)_n$ a sequence of non-negative times converging to t. Show that the r.v.'s $\limsup_n X_{t_n}$ and $\liminf_n X_{t_n}$ are measurable with respect to $\mathcal{F}_{t^+} = \cap_{s>t}\mathcal{F}_s$; in particular, the set $\{\exists \lim_n X_{t_n} = X_n\}$ is in $\mathcal{F}_{t^+}$.
Is it true that the r.v. $\limsup_{r\to t} X_r$ is $\mathcal{F}_{t^+}$-measurable? What if X is right-continuous (or left-continuous)?

3 Gaussian r.v.'s

Exercise 9. Let $(X_n)_n$ be a family of d-dimensional Gaussian r.v.'s, converging in L^2 to a r.v. X. Show that X is Gaussian and find its mean and convariance matrix. A stronger result holds: the thesis remains true if the convergence is only in law (see Baldi 0.16). [Recall that Y is Gaussian with mean m and cocariance matrix A if and only if its characteric function is $\hat{Y}(\xi) = \exp[im \cdot \xi - \frac{1}{2}\xi \cdot A\xi]$.]

Exercise 10. Let X be a $\mathbb{R}^d$-valued non-degenerate Gaussian r.v., with mean m and covariance matrix A. Find an expression for its moments. Notice that it is enough to consider the case when $m = 0$ and $A = I_d$ (the identity matrix).

4 Brownian motion

Exercise 11 (Svolto). Prove that the function $C(t, s) = s \wedge t = \min\{s, t\}$ is a covariance function.

Exercise 12. Show that a Brownian motion is unique in law.

Exercise 13 (Svolto). Verify the scaling invariance of the Brownian motion: if W is a real Brownian motion, with respect to a filtration $\mathcal{F}$, then

1. for every $r > 0$, $(W_{s+r} - W_r)_t$ is a Brownian motion with respect to $(\mathcal{F}_{s+r})_s$;
2. $-W$ is a Brownian motion with respect to $(\mathcal{F}_s)_s$;
3. for every $c > 0$, $(\frac{1}{\sqrt{c}} W_{cs})_s$ is a Brownian motion with respect to $(\mathcal{F}_{cs})_s$;
4. the process Z, defined as $Z_0 = 0$, $Z_s = sW_{1/s}$ for $s > 0$, is a Brownian motion with respect to its natural completed filtration.

A remark on property 3: if we put $s = t/c$, 3 implies that $\tilde{W}_s := \frac{1}{\sqrt{c}} W_t$ is a Brownian motion; in other words, we can say that the space scales as the square root of the time, in law. Think about the links between this and Hölder and no BV properties of the trajectories.

Exercise 14 (Svolto). Let W be a real Brownian motion. Show that, for a.e. trajectory,

$$\lim_{t \to +\infty} \frac{W_t}{t} = 0. \tag{1}$$

Hint: use Exercise 13.

Exercise 15. Let W be a real Brownian motion. For $T > 0$, ω in Ω, define the random set $S_T(\omega) = \{t \in [0,T] | W_t(\omega) > 0\}$. Show that the law of the r.v.

$$\frac{\mathcal{L}^1(S_T)}{T} \tag{2}$$

is independent of T ($\mathcal{L}^1$ is the 1-dimensional Lebesgue measure). This law is in fact the arcsine law: $P(\mathcal{L}^1(S_T) \leq \alpha T) = \frac{2}{\pi} \arcsin(\sqrt{\alpha})$.

Exercise 16. Let W be a real Brownian motion. Prove that, for a.e. ω in Ω, the trajectory $W(\omega)$ is not monotone on any interval $[a,b]$, with $a < b$; in particular, there exists a dense set S in $[0,+\infty[$ of local maximum points for $W(\omega)$.

Exercise 17. Let X be a real process. Given an interval $[a,b]$ of $[0,+\infty[$, call $M_{[a,b]} = \sup_{t \in [a,b]} X_t$. Prove that the law of M depends only on the law of X.
Now suppose $X = W$ is a real Brownian motion. Prove the following facts:

- If $b > a > 0$, the law of $M_{[a,b]}$ is diffuse (i.e. absolutely continuous with respect to Lebesgue measure).
- For a.e. ω in Ω, the trajectory $W(\omega)$ assumes different maxima on every couple of (non-trivial) compact disjoint intervals with rational extrema; in particular, every local maximum for $W(\omega)$ is strict.

Exercise 18. Let W be a real Brownian motion. Prove that, if A is a negligible set of $\mathbb{R}^d$ (with respect to $\mathcal{L}^1$), then, for a.e. ω, the random set $C(\omega) = \{t \in [0,T] | W_t(\omega) \in A\}$ is negligible (again with respect to $\mathcal{L}^1$).

Exercise 19 (Difficile). Show that there exists a sequence $(t_k)_k$ of positive times, converging (decreasingly) to 0, such that the family of r.v.'s

$$\frac{1}{n} \sum_{k=1}^{n} \operatorname{sign}(W_{t_k}) \tag{3}$$

converges to 0 in L^2 and a.s..
Hint: use independence of the increments, scaling invariance and Borel-Cantelli lemma.

Exercise 20 (Svolto). Let W be a real Brownian motion; let $\mathcal{F}$ be the natural completed filtration. We remind that, by Blumenthal 0-1 law, $\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$. Show that, for every sequence t_n, decreasing to 0, the events $\limsup_n\{W_{t_n} > 0\}$, $\liminf_n\{W_{t_n} < 0\}$ occur with probability 1. Deduce that, on every interval $[0, \delta]$, $\delta > 0$, W passes through 0 infinite times, with probability 1.

Exercise 21. For a function $f : \mathbb{R} \to \mathbb{R}$, we define the upper and lower derivative in t as

$$D_t^+ f = \limsup_{h\to 0} \frac{f(t+h) - f(t)}{h}, \quad D_t^- f = \liminf_{h\to 0} \frac{f(t+h) - f(t)}{h}. \tag{4}$$

Let W be a real Brownian motion, let $\mathcal{F}$ be the natural completed filtration; we remind that, by Blumenthal 0-1 law, $\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$. Fix $t \geq 0$. Show that, for every real C, the event $\{D_t^+ W \leq C\}$ is in $\mathcal{F}_t$. Prove, possibly without fine non-differentiability results, that the event $\{D_t^+ W = +\infty\}$ occurs with probability 1; analogously, $\{D_t^- W = -\infty\}$ has probability 1.

Exercise 22 (Svolto in parte). A modified OU process is a real process X, which has $\mathbb{R}$ as time set, has continuous trajectories, is Gaussian centered with covariance function $Cov(s, t) = E[X_s X_t] = \exp[-|t - s|]$.
Suppose that W is a (continuous) real Brownian motion. Prove that $X_t = e^{-t} W_{e^{2t}}$ is a modified OU process.
Suppose that X is a modified OU process. Prove that $W_t = t^{1/2} X(\frac{1}{2}\log t)$ is a (continuous a.s.) Brownian motion.

Exercise 23. Define the (1-dimensional) Brownian bridge as the Gaussian process $(X_t)_t$, $t \in [0, 1]$, with mean function $E[X_t] = 0$ and covariance function $E[X_s X_t] = s \wedge t - st$ ($s \wedge t = \min\{s, t\}$); it exists and is unique in law (why?). Prove the following facts:

- If W is a real Brownian motion, then $X_t = W_t - tW_1$ is a Brownian bridge.
- If X is a Brownian bridge and ξ is a $\mathcal{N}(0, 1)$ real r.v. independent of X, then $W_t = X_t + t\xi$ is a Brownian motion.

The second statement tells that, in some sense, a Brownian bridge is a Brownian motion, given $W_1 = 0$.

Exercise 24. A d-dimensional Brownian motion, with respect to a filtration $\mathcal{F} = (\mathcal{F}_t)_t$, is a process B, adapted to $\mathcal{F}$, with values in $\mathbb{R}^d$, such that:

- $B_0 = 0$ a.s.;
- B has independent increments (with respect to the filtration $\mathcal{F}$);
- for every $s < t$, the law of $B_t - B_s$ is centered Gaussian with covariance matrix $(t - s)I_d$ (I_d is the $d \times d$ identity matrix);
- B has continuous trajectories.

In case $\mathcal{F}$ is the natural completed filtration of B, we call B a standard Brownian motion. Show that a d-dimensional process X is a standard Brownian motion if and only if its components X_j's are real independent standard Brownian motions.

5 Stopping times

Exercise 25 (Svolto). Let X be a continuous process, with values in a metric space E, adapted to a right-continuous completed filtration $\mathcal{F}$; let G, F be an open, resp. closed set in E. Define $\tau_G = \inf\{t \geq 0 | X_t \notin G\}$, $\rho_F = \inf\{t \geq 0 | X_t \in F\}$; notice that $\rho_F = \tau_{F^c}$. Prove that τ_G, ρ_F are stopping times with respect to $\mathcal{F}$.
Convention: unless otherwise stated, if the set $\{t \geq 0 | X_t \notin G\}$ is empty, we define $\tau_G = +\infty$; an analogous convention will be used in the sequel for similar cases.

Exercise 26 (Svolto in parte). With the notation of Exercise 25, show that the laws of τ_G and of ρ_F depend only on the law of X. Prove also that the laws of (τ_G, X) and (ρ_F, X), as r.v.'s with values in $[0, +\infty] \times C([0,T]; E)$, depend only on the law of X.

Exercise 27. Let X be a real process, adapted to $\mathcal{F}$, and let τ be a random time, such that $\{\tau < t\}$ is in $\mathcal{F}_t$ for every t. Show that τ is a stopping time with respect to the augmented filtration $\mathcal{F}^+ = (\mathcal{F}_{t^+})_t$, where $\mathcal{F}_{t^+} = \cap_{s>t} \mathcal{F}_s$.

Exercise 28 (Consigliato). With the notation of Exercise 25, call $\tau_G^{(j)}$ the time of the j-th exit from G; more precisely, $\tau_G^{(1)} = \tau_G$, $\tau_G^{(j+1)} = \inf\{t > \tau_G^{(j)} | X_t \notin G\}$. Prove that, for every j, $\tau_G^{(j)}$ is a stopping time with respect to $\mathcal{F}$.
Call τ_G^{last} the last (possibly infinite) time when X exits G, i.e. $\tau_G^{last} = \inf\{t \geq 0 | X_s \notin G \ \forall s > t\}$. Find an example of X, $\mathcal{F}$ and G such that τ_G^{last} is not a stopping time.

Exercise 29. Let $(\mathcal{F}_t)_t$ be a filtration (on a probability space $(\Omega, \mathcal{A}, P)$) and let $X = (X_t)_t$ be a real-valued progressively measurable process with respect to $(\mathcal{F}_t)_t$. Let τ be a finite stopping time with respect to $(\mathcal{F}_t)_t$ and let $\mathcal{F}_\tau$ be the associated σ-algebra, defined as

$$\mathcal{F}_\tau := \{A \in \mathcal{F}_\infty | A \cap \{\tau \leq t\} \in \mathcal{F}_t \ \forall t\} \tag{5}$$

(where $\mathcal{F}_\infty$ is the σ-algebra generated by all the $\mathcal{F}_t$'s). Define $X_\tau : \Omega \to \mathbb{R}$ as $X_\tau(\omega) := X_{\tau(\omega)}(\omega)$. Prove that X_τ is measurable with respect to $\mathcal{F}_\tau$.

Exercise 30. Prove the stopping theorem for Brownian motion: If W is a real Brownian motion motion with respect to a filtration $(\mathcal{F}_t)_t$ and τ is a finite stopping time (with respect to the same filtration), then $(W_{t+\tau} - W_\tau)_t$ is a standard Brownian motion, independent of $\mathcal{F}_\tau$.

Exercise 31. Let W be a real Brownian motion. Show that, for every x in $\mathbb{R}$, for every $\delta > 0$, the event

$$\{\omega | \forall t \geq 0, \text{ if } W_t(\omega) = x, \text{ then } W(\omega) \text{ passes through } x \text{ infinite times in } [t, t+\delta]\} \tag{6}$$

has probability 1.
Hint: remember Exercise 20 and use stopping theorem for Brownian motion.

Exercise 32. Let X be a continuous process with values in $\mathbb{R}^2$, such that $X_0 = 0$ and the law of X is invariant under rotation (i.e. the processes $(RX_t)_t$ and $(X_t)_t$ are equivalent, for every orthognonal matrix R). Let $\tau = \inf\{t \geq 0 | |X_t| = 1\}$ and suppose $\tau < +\infty$ a.s. (see the convention in Exercise 25). Show that τ and X_τ are independent r.v.'s and that

the law of X_τ is λ_{S^1}, the renormalized Lebesgue measure on the sphere S^1.
We recall that λ is the unique probability measure on S^1 with is invariant under rotation and satisfies $\lambda([c\alpha, c\beta]) = c\lambda([\alpha, \beta])$ for every $c > 0$, α, β angles.
Find examples of $\mathbb{R}^2$-valued continuous processes X (with $X_0 = 0$) such that:

- τ, X_τ are independent, but the law of X_τ is not λ_{S^1};
- the law of X_τ is λ_{S^1}, but τ and X_τ are not independent;
- for every $t > 0$, $P(X_t = 0) = 0$ and the law of $\frac{X_t}{|X_t|}$ is λ_{S^1}, but the law of X_τ is not λ_{S^1}.

6 Martingales: examples, stopping theorem, Doob and maximal inequalities

Exercise 33. Let M be a (continuous- or discrete-time) martingale, with respect to a filtration $\mathcal{F}$; let $\mathcal{G}$ be a subfiltration of $\mathcal{F}$ (i.e. $\mathcal{G}_t \subseteq \mathcal{F}_t$ for every t), such that M is still adapted to $\mathcal{G}$. Verify that M is a martingale with respect to $\mathcal{G}$.

Exercise 34. Let $(\xi_j)_j$ be a sequence of i.i.d. r.v.'s. Let f be a measurable function such that $E[|f(\xi_1)|] < +\infty$. Prove that the following processes are martingales:

- $X_n = \sum_{j=1}^n f(\xi_j) - nE[f(\xi_1)]$;
- $Y_n = E[f(\xi_1)]^{-n} \prod_{j=1}^n f(\xi_j)$, if $E[f(\xi_1)] \neq 0$.

Exercise 35 (Svolto). Let X be a process with independent increments (with respect to a filtration $\mathcal{F}$). Prove that $X_t - E[X_t]$ is a martingale (with respect to $\mathcal{F}$).
Let Y be a process with centered independent increments. Prove that $Y_t^2 - E[Y_t^2]$ is a martingale.

Exercise 36 (Consigliato per la prima parte). Let W be a Brownian motion. Prove that W_t, $W_t^2 - t$, $\exp[\lambda W_t - \frac{1}{2}\lambda^2 t]$, λ in $\mathbb{C}$, are martingales.
Let $X = (X_t)$ be a continuous process, adapted to $\mathcal{F}$, with $X_0 = 0$. Suppose that $\exp[i\lambda X_t + \frac{1}{2}\lambda^2 t]$ is a martingale, for every real λ. Prove that X is a Brownian motion with respect to $\mathcal{F}$.

Exercise 37 (Svolto (senza dimostrazione per la seconda parte), enunciato corretto). Let M be a discrete-time martingale, let τ be a finite stopping time. Prove that the process M^τ, defined by $M_n^\tau = M_{\tau\wedge n}$, is a martingale (with respect to the same filtration of M).
Let X be a continuous-time martingale with continuous trajectories, let τ be a stopping time (with values in $[0, +\infty]$). Prove that M^τ, defined as $M_t^\tau = M_{\tau\wedge t}$, is a martingale (again with respect to the same filtration).

Exercise 38 (Svolto). Let W be a Brownian motion; for $a, b > 0$, call $\tau_{a,b} = \inf\{t \geq 0 | W_t \notin] - a, b[\}$. Compute $E[\tau_{a,b}]$ and $P\{B_{\tau_{a,b}} = b\}$.

Exercise 39 (Svolto). Let W be a Brownian motion; for $a > 0$, call $\tau_a = \inf\{t \geq 0 | W_t = a\}$. Prove that τ is finite a.s. and that $E[\exp[-\lambda^2\tau/2]] = e^{-\lambda a}$ for every $\lambda > 0$. Prove that $E[\tau] = +\infty$.
Hint: For the first part, consider the martingale $M_t = \exp[\lambda W_t - \lambda^2 t/2]$ and let λ go to 0; for the second part, use Exercise 38.

Exercise 40 (Consigliato). Let W be a real Brownian motion; for $\gamma > 0$, $a > 0$, let $\rho_{\gamma,a} = \inf\{t \geq 0 | W_t = a + \gamma t\}$. Prove that $P(\rho_{\gamma,a} < +\infty) = \exp[-2\gamma a]$.
Hint: consider the martingale $M_t = \exp[2\gamma W_t - 2\gamma^2 t]$ and remember the behaviour of the Brownian motion at infinity.

Exercise 41 (Svolto). Let X be a continuous sub-martingale, let $h : \mathbb{R} \to \mathbb{R}$ be a positive convex nondecreasing function, $\theta > 0$. Prove that

$$P(\sup_{[0,T]} X_t \leq \lambda) \leq E[h(\theta X_T)]/h(\theta\lambda). \tag{7}$$

If X is a Brownian motion, with $h(x) = e^x$ and taking the infimum over $\theta > 0$, we get the exponential bound for the Brownian motion.

Exercise 42. Let $(\xi_j)_j$ a family of i.i.d. Bernoulli r.v.'s, with $P(\xi_1 = 1) = p$ and $P(\xi_1 = -1) = q = 1-p$; suppose that $p < q$. Let $S_n = \sum_{j=1}^n \xi_j$ be the associated non-symmetric random walk, define $Z_n = (q/p)^{S_n}$.

1. Prove that Z is a martingale.
2. Prove that $P(\sup_n S_n \geq k) \leq (p/q)^k$ and deduce that $E[\sup_n S_n] \leq p/(q-p)$.

Exercise 43. Let $M = (M_n)_n$ be a nonnegative supermartingale (with respect to some filtration $(\mathcal{F}_t)_t$), with M_0 constant and such that $M_{n+1} - M_n \leq C$ for every n, for some $C > 0$. Fix $\lambda > M_0$ and define $\tau = \inf\{n | M_n > \lambda\} \wedge N$ (stopping time ≥ 1, with associated σ-algebra $\mathcal{F}_\tau$).

1. Prove that, on $\{M_\tau > \lambda\}$, $M_\tau \leq C + \lambda$ and deduce that $P\{\sup_{n=0,\dots N} M_n > \lambda\} \geq \frac{1}{C+\lambda}\int_{M_\tau > \lambda} M_\tau dP$.
2. Prove that the event $\{M_\tau > \lambda\}$ belongs to $\mathcal{F}_\tau$.
3. Prove that $P\{\sup_{n=0,\dots N} M_n > \lambda\} \geq \frac{1}{C+\lambda}\int_{M_\tau > \lambda} M_N dP$.

Exercise 44 (Svolto per i primi due punti). Let $(\xi_n)_n$ be i.

Exercise 45. Let W be a real Brownian motion. Prove the following facts:

- For a.e. ω in Ω, the trajectory $W(\omega)$ is not monotone on any interval $[a, b]$, with $a < b$; in particular, there exists a dense set S in $[0, +\infty[$ of local maximum points for $W(\omega)$.
- For a.e. ω in Ω, the trajectory $W(\omega)$ assumes different maxima on every couple of (non-trivial) compact disjoint intervals with rational extrema; in particular, every local maximum for $W(\omega)$ is strict.

i.d. r.v.'s with $P\{\xi_0 = 0\} = P\{\xi_0 = 1\} = 1/2$. Define the process Y as $Y_0 = 0$, $Y_{n+1} = Y_n + 2^{-n-1}\xi_n$, let $\mathcal{F}_n$ the filtration generated by Y (i.e. $\mathcal{F}_n = \sigma\{Y_0, \dots Y_n\} = \sigma\{\xi_0, \dots \xi_{n-1}\}$).
Let f be a bounded measurable function, define the process X as

$$X_n = \frac{f(Y_n + 2^{-n}) - f(Y_n)}{2^{-n}}. \tag{8}$$

1. Prove that $E[f(Y_{n+1})|\mathcal{F}_n] = \frac{1}{2}(f(Y_n + 2^{-n-1}) + f(Y_n))$.
2. Prove that X is a martingale (with respect to $\mathcal{F}$).
3. Suppose that f is nondecreasing. Prove that X converges a.s..
4. Suppose that f is a Lipschitz function (i.e. $|f(x) - f(y)| \leq C|x - y|$). Prove that X converges a.s. and in L^p, for every finite $p \geq 1$.

Exercise 46 (Difficile). A monkey types randomly a capital letter; the sequence of letters is represented by a sequence of i.i.d. r.v.'s with uniform distribution on the set of 26 letters. Prove that the mean time required to write ABRACADABRA is $26^{11} + 26^4 + 26$. Hint: Consider a proper martingale ....

7 Martingales: convergence and Doob-Meyer decomposition

Exercise 47 (Svolto). Let $(X_j)_j \in \mathbb{N}^+$ be a family of independent r.v.'s in L^2, with $E[X_j] = 0$ for every j and $\sum_{j=1}^{\infty} E[X_j^2] < +\infty$. Define $\mathcal{F}_\backslash = \sigma\{X_1, \dots X_n\}$ and $M_n := \sum_{j=1}^{n} X_j$. Prove that $(M_n)_n$ is a martingale (with respect to $(\mathcal{F})_n$) and that M_n converges a.s. and in L^2, for $n \to +\infty$.

Exercise 48 (Svolto). Let W be a real Brownian motion, define the martingale M as $M_t = \exp[\lambda W_t - \lambda^2 t/2]$. Prove that M converges a.s. to 0, but does not converge in L^1.

Exercise 49 (Svolto). Let M be a continuous martingale in L^2, with $M_0 = 0$. We remind that, since the process $A = \langle M \rangle$ is nonnegative nondecreasing, it converges to some finite or infinite limit, as $t \to +\infty$. Prove the following dichotomy:

- on $\{\lim_{t\to+\infty} A_t < +\infty\}$, M converges a.s. as $t \to +\infty$;
- on $\{\lim_{t\to+\infty} A_t = +\infty\}$, it holds for a.e. ω: for every $a \geq 0$, the trajectory $|M(\omega)|$ reaches a at least one time; in particular, a.e. trajectory does not converge.

Hint: use convergence results for the martingale N, stopped at time $\sigma_a := \inf\{t \geq 0 | A_t = a\}$ or at time $\tau_a := \inf\{t \geq 0 || M_t| = a\}$.