Exercises in stochastic analysis

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The exercises with a P are those which have been done (totally or partially) in the previous lectures; the exercises with an N could be done (totally or partially) in the next classes. Of course, this is a pre-selection, one more selection and changes are possible.

1 Laws and marginals

Exercise 1 (P). *Let τ be a random variable (r.v.) with values in $[0,+\infty[$, define the (continuous-time) waiting process X as $X_t = 1_{\tau<t}$. Compute the m-dimensional distributions of X, for any m integer.*

Exercise 2 (P). *Exhibit two processes, which are not equivalent but have the same 1- and 2-dimensional distributions. Exhibit a family of compatible 2-dimensional distribions, which does not admit a process having those distributions as 2-dimensional marginals.*

Exercise 3. *Let $(F,\mathcal{F})$ be a measurable space, G a subset (not necessarily measurable) of F, define the restriction of $\mathcal{F}$ to G as $\mathcal{F}_{|G} := \{A\cap G | A\in\mathcal{F}\}$; it is easy to see that $\mathcal{F}_{|G}$ is a σ-algebra on G.*
Now consider $(E,\mathcal{E})$ a measurable space (morally, the state space for some continuous process), call $C = C([0,T];E)$. Show that $\mathcal{B}(C) = (\mathcal{E}^{[0,T]})_{|C}$. This allows to define the law of a continuous process on C, directly from the definition of law for a generic process.

2 Trajectories and filtrations

Exercise 4. *Let X be a continuous process. Prove that its natural filtration $(\mathcal{F}_t)_t$ ($\mathcal{F}_t = \sigma(X_s | s\le t)$) is left-continuous, that is $\mathcal{F}_t = \bigvee_{s<t}\mathcal{F}_s$.*

Exercise 5. *Let X, Y be two processes, a.e. continuous and modifications one of each other. Show that they are indistinguishable.*
Let X be a process, such that a.e. trajectory is continuous. Show that there exists a process U, indistinguishable of X, which is everywhere continuous, i.e. every trajectory is continuous.

Exercise 6. *Let X be a progressively measurable bounded process. Show that the process $(Y_t = \int_0^t X_s ds)_t$ is progressively measurable (with respect to the same filtration).*

Exercise 7. *Let X, Y be two processes, modifications one of each other. Suppose that X is adapted to a filtration $\mathcal{F} = (\mathcal{F}_t)_t$ and that $\mathcal{F}_0$ is completed, that is, contains all the P-negligible sets. Prove that also Y is adapted to $\mathcal{F}$.*

Exercise 8 (P)**.** *Let X be a process, adapted to a filtration $\mathcal{F}$. Fix $t \geq 0$ and $(t_n)_n$ a sequence of non-negative times converging to t. Show that the r.v.'s $\limsup_n X_{t_n}$ and $\liminf_n X_{t_n}$ are measurable with respect to $\mathcal{F}_{t+} = \cap_{s>t}\mathcal{F}_s$; in particular, the set $\{\exists \lim_n X_{t_n} = X_n\}$ is in $\mathcal{F}_{t+}$.*
Is it true that the r.v. $\limsup_{r\to t} X_r$ is $\mathcal{F}_{t+}$-measurable? What if X is right-continuous (or left-continuous)?

3 Gaussian r.v.'s

Exercise 9 (P)**.** *Let $(X_n)_n$ be a family of d-dimensional Gaussian r.v.'s, converging in L^2 to a r.v. X. Show that X is Gaussian and find its mean and convariance matrix. A stronger result holds: the thesis remains true if the convergence is only in law (see Baldi 0.16). [Recall that Y is Gaussian with mean m and cocariance matrix A if and only if its characteric function is $\hat{Y}(\xi) = \exp[im\cdot\xi - \frac{1}{2}\xi\cdot A\xi]$.]*

Exercise 10. *Let X be a $\mathbb{R}^d$-valued non-degenerate Gaussian r.v., with mean m and covariance matrix A. Find an expression for its moments.*

4 Conditional expectation and conditional law

Exercise 11 (P)**.** *Prove the conditional versions of the following theorems: Fatou, Lebesgue, Jensen.*

Exercise 12. *Let $X_1, \dots X_n$ a family of i.i.d. r.v.'s, call $S = \sum_{k=1}^n X_k$. Prove that $E[X_k|S]$ is independent of k and so is $\frac{1}{n}S$.*

Exercise 13. *Let X be a r.v., $\mathcal{G}$, $\mathcal{H}$ be two filtrations such that X is independent of $\mathcal{H}$. Is it true that $E[X|\mathcal{G}\bigvee\mathcal{H}] = E[X|\mathcal{G}]$?*

Exercise 14 (P)**.** *Let X, Y be two r.v.'s with values in $\mathbb{R}^d$. We know, from a measurability theorem by Doob (see the notes and Baldi, Chapter 0), that, for every Borel bounded function f on $\mathbb{R}^d$, there exists a measurable function $g = g^f$ on $\mathbb{R}^d$, such that $E[f(X)|Y] = g(Y)$ a.s.. Notice that the statement remains true if we change g outside a negligible set with respect to the law of Y.*
Prove that, for every f, g^f depends only on the law of (X, Y). Conversely, prove that the law of (X, Y) depends only on the law of Y and such a family $(g^f)_f$.

Exercise 15 (P)**.** *Here is an example where the g of the previous exercise (and so the conditional expectiation) can be computed. Let X, Y be two r.v.'s with values in $\mathbb{R}^d$, suppose that (X, Y) has an absolutely continuous law (with respect to the Lebesgue measure), call ρ_Y, $\rho_{X,Y}$ the densities of the laws resp. of Y, (X, Y) (which is the relation* *between them?). Prove that, for every Borel bounded function f on $\mathbb{R}^d$, a version of $E[f(X)|Y] = g(Y)$ is given by*

$$g(y) = \int_{\mathbb{R}^d} f(x)h(x,y)dx, \tag{1}$$

$$h(x,y) = \frac{\rho_{X,Y}(x,y)}{\rho_Y(y)} 1_{\rho_Y>0}(y). \tag{2}$$

Notice that the statement remains true if we change h outside the set $\{\rho_Y > 0\}$.

Exercise 16. *With the notation of the previous exercise, compute h when (X,Y) is a non-degenerate Gaussian r.v. with mean m and covariance matrix A.*
Notice that, even if h is defined up to $Y_\# P$-negligible sets (and so up to Lebesgue-negligible sets, since Y is Gaussian non-degenerate), the expression (2) gives the unique continuous representative of h (we will call it h as well). Notice also that, for fixed y, $h(\cdot,y)$ is a Gaussian density. For such h, we will denote the corresponding g^f as $g^f(y) = E[f(X)|Y=y]$.

5 Markov processes

Exercise 17. *Show that the Markow property, with respect to the natural filtration, depends only on the law of a process: if X is a Markov process (with respect to $\mathcal{F}_t = \sigma(X_s|s \le t)$) and Y has the same law of X, then Y is also Markov (with respect to $\mathcal{G}_t = \sigma(Y_s|s \le t)$). Show also that, if p is a transition function for X, then p is a transition function also for Y.*

Exercise 18. *Show that, if X is a Markov process with respect to a filtration $\mathcal{F}$, then it is Markov also with respect to a subfiltration $\mathcal{G}$ (such that X is adapted to $\mathcal{G}$).*

Exercise 19 (P)**.** *Let S_n be a symmetric random walk, call $M_n = \max_{0\le j\le n} S_j$. Show that M is not a Markov process.*

Exercise 20. *Find a Markov process X, such that $|X|$ is not Markov (with respect to the same filtration of X or even the natural filtration of $|X|$). Can you guess a sufficient condition, for example on the joint law of $(X_s, |X_t|)$ (for $s < t$), which makes $|X|$ Markov?*

Exercise 21 (P)**.** *Let X be a process with independent increments. Show that X is a Markov process.*

Exercise 22. *Let X be a process with independent increments; call μ_0 the law of X_0, $\mu_{s,t}$ the law of $X_t - X_s$, for $s < t$. Show that these laws determines uniquely the law of X and that, for every $s < r < t$, it holds $\mu_{s,t} = \mu_{s,r} * \mu_{r,t}$, that is*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x+y)\mu_{s,r}(dx)\mu_{s,r}(dy) = \int_{\mathbb{R}^d} f(z)\mu_{s,t}(dz). \tag{3}$$

Conversely, let μ_0, $\mu_{s,t}$, $0 \leq s < t$, be a family of probability measures on $\mathbb{R}^d$ such that, for every $s < r < t$, it holds $\mu_{s,t} = \mu_{s,r} * \mu_{r,t}$. Show that there exists a process X such that the law of X_0 is μ_0 and, for every $s < t$, the law of $X_t - X_s$ is $\mu_{s,t}$. Is there a filtration which makes X a process with independent increments?

Exercise 23. *Let X be a process with independent increments, with values in $\mathbb{R}^d$. Suppose that, for some $s \geq 0$, X_s has a law which is absolutely continuous with respect to the Lebesgue measure (resp. non atomic). Prove that, for every $t \geq s$, X_t has an absolutely continuous (resp. non atomic) law.*

6 Brownian motion

Exercise 24. *Show that a Brownian motion is unique in law.*

Exercise 25 (P)**.** *Verify the scaling invariance of the Brownian motion: if W is a real Brownian motion, with respect to a filtration $\mathcal{F}$, then*

1. *for every $r > 0$, $(W_{s+r} - W_r)_t$ is a Brownian motion with respect to $(\mathcal{F}_{s+r})_s$;*
2. *$-W$ is a Brownian motion with respect to $(\mathcal{F}_s)_s$;*
3. *for every $c > 0$, $(\frac{1}{\sqrt{c}} W_{cs})_s$ is a Brownian motion with respect to $(\mathcal{F}_{cs})_s$;*
4. *the process Z, defined as $Z_0 = 0$, $Z_s = sW_{1/s}$ for $s > 0$, is a Brownian motion with respect to its natural completed filtration.*

A remark on property 3: if we put $s = t/c$, 3 implies that $\tilde{W}_s := \frac{1}{\sqrt{c}} W_t$ is a Brownian motion; in other words, we can say that the space scales as the square root of the time, in law. Think about the links between this and Hölder and no BV properties of the trajectories.

Exercise 26 (P)**.** *Let W be a real Brownian motion. Show that, for a.e. trajectory,*

$$\lim_{t \to 0} \frac{W_t}{t} = 0. \tag{4}$$

Hint: use Exercise 25.

Exercise 27 (P)**.** *Let W be a real Brownian motion. For $T > 0$, ω in Ω, define the random set $S_T(\omega) = \{t \in [0, T] | W_t(\omega) > 0\}$. Show that the law of the r.v.*

$$\frac{\mathcal{L}^1(S_T)}{T} \tag{5}$$

is independent of T ($\mathcal{L}^1$ is the 1-dimensional Lebesgue measure). This law is in fact the arcsine law: $P(\mathcal{L}^1(S_T) \leq \alpha T) = \frac{2}{\pi} \arcsin(\sqrt{\alpha})$.

Exercise 28. *Let W be a real Brownian motion. Prove the following facts:*

- *For a.e. ω in Ω, the trajectory $W(\omega)$ is not monotone on any interval $[a, b]$, with $a < b$; in particular, there exists a dense set S in $[0, +\infty[$ of local maximum points for $W(\omega)$.*

- *For a.e. ω in Ω, the trajectory $W(\omega)$ assumes different maxima on every couple of (non-trivial) compact disjoint intervals with rational extrema; in particular, every local maximum for $W(\omega)$ is strict.*

Exercise 29. *Let W be a real Brownian motion. Prove that, if A is a negligible set of $\mathbb{R}^d$ (with respect to $\mathcal{L}^1$), then, for a.e. ω, the random set $C(\omega) = \{t \in [0, T] | W_t(\omega) \in A\}$ is negligible (again with respect to $\mathcal{L}^1$).*

Exercise 30. *Let W be a real Brownian motion. Show that, on every interval $[0, \delta]$, $\delta > 0$, W passes through 0 infinite times, with probability 1.*

Exercise 31. *A d-dimensional Brownian motion, with respect to a filtration $\mathcal{F} = (\mathcal{F}_t)_t$, is a process B, adapted to $\mathcal{F}$, with values in $\mathbb{R}^d$, such that:*

- *$B_0 = 0$ a.s.;*

- *B has independent increments (with respect to the filtration $\mathcal{F}$);*

- *for every $s < t$, the law of $B_t - B_s$ is centered Gaussian with covariance matrix $(t - s)I_d$ (I_d is the $d \times d$ identity matrix);*

- *B has continuous trajectories.*

In case $\mathcal{F}$ is the natural completed filtration of B, we call B a standard Brownian motion. Show that a d-dimensional process X is a standard Brownian motion if and only if its components X_j's are real independent standard Brownian motions.

Exercise 32. *Let W a real Brownian motion. Prove that, for every $T > 0$, $\alpha < 1/2$, there exists $c > 0$ such that the event $\{W_t \leq ct^\alpha, \forall t \in [0, T]\}$ occurs with positive probability.*
Hint: prove first than, for any fixed c, there exists a (small) $T > 0$, depending only on c, α and the law of W, such that the above event is not negligible.

7 Stopping times

Exercise 33 (P)**.** *Let X be a continuous process, with values in a metric space E, adapted to a right-continuous completed filtration $\mathcal{F}$; let G, F be an open, resp. closed set in E. Define $\tau_G = \inf\{t \geq 0 | X_t \notin G\}$, $\rho_F = \inf\{t \geq 0 | X_t \in F\}$; notice that $\rho_F = \tau_{F^c}$. Prove that τ_G, ρ_F are stopping times with respect to $\mathcal{F}$.*
Show that the laws of τ_G and of ρ_F depend only on the law of X. Prove also that the laws of (τ_G, X) and (ρ_F, X), as r.v.'s with values in $[0, +\infty] \times C([0, T]; E)$, depend only on the law of X.
Convention: unless otherwise stated, if the set $\{t \geq 0 | X_t \notin G\}$ is empty, we define $\tau_G = +\infty$; an analogous convention will be used in the sequel for similar cases.

Exercise 34. *Let X be a real process, adapted to $\mathcal{F}$, and let τ be a random time, such that $\{\tau < t\}$ is in $\mathcal{F}_t$ for every t. Show that τ is a stopping time with respect to the augmented filtration $\mathcal{F}^+ = (\mathcal{F}_{t+})_t$, where $\mathcal{F}_{t+} = \cap_{s>t}\mathcal{F}_s$.*

Exercise 35. *Prove the stopping theorem for Brownian motion: If W is a real Brownian motion motion with respect to a filtration $(\mathcal{F}_t)_t$ and τ is a finite stopping time (with respect to the same filtration), then $(W_{t+\tau} - W_\tau)_t$ is a standard Brownian motion, independent of $\mathcal{F}_\tau$.*

Exercise 36. *Let X be a continuous process with values in $\mathbb{R}^2$, such that $X_0 = 0$ and the law of X is invariant under rotation (i.e. the processes $(RX_t)_t$ and $(X_t)_t$ are equivalent, for every orthognonal matrix R). Let $\tau = \inf\{t \geq 0 | |X_t| = 1\}$ and suppose $\tau < +\infty$ a.s. (see the convention in Exercise 33). Show that τ and X_τ are independent r.v.'s and that the law of X_τ is λ_{S^1}, the renormalized Lebesgue measure on the sphere S^1.*
We recall that λ is the unique probability measure on S^1 with is invariant under rotation and satisfies $\lambda([c\alpha, c\beta]) = c\lambda([\alpha,\beta])$ for every $c > 0$, α, β angles.
Find examples of $\mathbb{R}^2$-valued continuous processes X (with $X_0 = 0$) such that:

- *τ, X_τ are independent, but the law of X_τ is not λ_{S^1};*
- *the law of X_τ is λ_{S^1}, but τ and X_τ are not independent;*
- *for every $t > 0$, $P(X_t = 0) = 0$ and the law of $\frac{X_t}{|X_t|}$ is λ_{S^1}, but the law of X_τ is not λ_{S^1}.*

8 Martingales: examples, stopping theorem, Doob and maximal inequalities

Exercise 37. *Let M be a (continuous- or discrete-time) martingale, with respect to a filtration $\mathcal{F}$; let $\mathcal{G}$ be a subfiltration of $\mathcal{F}$ (i.e. $\mathcal{G}_t \subseteq \mathcal{F}_t$ for every t), such that M is still adapted to $\mathcal{G}$. Verify that M is a martingale with respect to $\mathcal{G}$.*

Exercise 38 (P). *Let $(\xi_j)_j$ be a sequence of i.i.d. r.v.'s. Let f be a measurable function such that $E[|f(\xi_1)|] < +\infty$. Prove that the following processes are martingales:*

- *$X_n = \sum_{j=1}^n f(\xi_j) - nE[f(\xi_1)]$;*
- *$Y_n = E[f(\xi_1)]^{-n} \prod_{j=1}^n f(\xi_j)$, if $E[f(\xi_1)] \neq 0$.*

Deduce that the following are martingales: ....

Exercise 39. *Find an example of a martingale which is not a Markov process.*

Exercise 40 (P). *Let X be a process with independent increments (with respect to a filtration $\mathcal{F}$). Prove that $X_t - E[X_t]$ is a martingale (with respect to $\mathcal{F}$).*
Let Y be a process with centered independent increments. Prove that $X_t^2 - E[X_t^2]$ is a martingale.

Exercise 41 (N)**.** *Let W be a Brownian motion. Prove that W_t, W_t^2-t, $\exp[\lambda W_t-\frac{1}{2}\lambda^2 t]$, λ in $\mathbb{C}$, are martingales.*
Let $X = (X_t)$ be a continuous process, adapted to $\mathcal{F}$, with $X_0 = 0$. Suppose that $\exp[\lambda W_t - \frac{1}{2}\lambda^2 t]$ is a martingale, for every real λ. Prove that X is a Brownian motion with respect to $\mathcal{F}$.

Exercise 42 (P)**.** *Let M be a discrete-time martingale, let τ be a finite stopping time. Prove that the process M^τ, defined by $M_n^\tau = M_{\tau\wedge n}$, is a martingale (with respect to the same filtration of M).*
Let X be a continuous-time martingale, let τ be a stopping time, suppose that the the hypotheses of the stopping theorem are satisfied. Prove that M^τ, defined as $M_t^\tau = M_{\tau\wedge t}$, is a martingale (again with respect to the same filtration).

Exercise 43 (N)**.** *Let W be a Brownian motion; for $a, b > 0$, call $\tau_{a,b} = \inf\{t \geq 0 | W_t \notin]-a, b[\}$. Compute $E[\tau_{a,b}]$ and $P\{B_{\tau_{a,b}} = b\}$.*

Exercise 44. *Let W be a real Brownian motion; for $\gamma > 0$, $a > 0$, let $\rho_{\gamma,a} = \inf\{t \geq 0 | W_t = a + \gamma t\}$. Prove that $P(\rho_{\gamma,a} < +\infty) = \exp[-2\gamma a]$.*
Hint: consider the martingale $M_t = \exp[2\gamma W_t - 2\gamma^2 t]$ and remember the behaviour of the Brownian motion at infinity.

Exercise 45 (N)**.** *Let X be a continuous sub-martingale, let $h : \mathbb{R} \to \mathbb{R}$ be a positive convex nondecreasing function, $\theta > 0$. Prove that*

$$P(\sup_{[0,T]} X_t \leq \lambda) \leq e^{-h(\theta\lambda)} E[\exp[h(\theta X_T)]]. \tag{6}$$

If X is a Brownian motion, taking $h(x) = e^x$ and making the infimum over $\theta > 0$, we get the exponential bound for the Brownian motion.

9 Martingales: convergence and Doob-Meyer decomposition

Exercise 46 (N)**.** *Let M be a continuous martingale in L^2, with $M_0 = 0$. We remind that, since the process $A = \langle M \rangle$ is nonnegative nondecreasing, it converges to some finite or infinite limit, as $t \to +\infty$. Prove the following dichotomy:*

- *on $\{\lim_{t\to+\infty} A_t < +\infty\}$, M converges a.s. as $t \to +\infty$;*
- *on $\{\lim_{t\to+\infty} A_t = +\infty\}$, it holds for a.e. ω: for every $a \geq 0$, the trajectory $|M(\omega)|$ reaches a at least one time; in particular, a.e. trajectory does not converge.*

Exercise 47. *Let $(M_t)_{t\geq 0}$ be a positive (i.e. $M_t \geq 0$) continuous supermartingale and let $T = \inf\{t \geq 0 : M_t = 0\}$. Prove that, if $T(\omega) < \infty$, then for any $t \geq T(\omega)$, $M_t(\omega) = 0$.*

10 Stochastic integrals

Through all this section, let $(B_t)_{t\geq 0}$ be a real continuous Brownian motion, starting from 0, defined on a filtered probability space $(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{F} = (\mathcal{F}_t)_{t\geq 0})$, where $\mathcal{F}$ satisfies the usual assumptions.

Exercise 48. *Prove that, for $0 \leq \alpha \leq \beta < \infty$ the space $M^2[\alpha,\beta]$ of ($\mathbb{P}\otimes\mathscr{L}$-equivalence classes of) progressively measurable real processes $(X_r)_{\alpha\leq r\leq\beta}$ such that*

$$\mathbb{E}\left[\int_\alpha^\beta |X_r|^2 \, dr\right] < \infty,$$

with the scalar product $\langle X, Y\rangle = E\left[\int_\alpha^\beta X_r Y_r dr\right]$, is a Hilbert space. (Hint: it is a closed subspace of $L^2\left(\Omega, \mathcal{A}\otimes\mathcal{B}\left([\alpha,\beta]\right), \mathbb{P}\otimes\mathscr{L}\right)$.)

Exercise 49. *(see Baldi, 6.10) Prove that, for $\alpha \leq \beta$, $M^2[\alpha,\beta] \ni H \mapsto \int_\alpha^\beta H_r dB_r$ is homogeneus with respect to the product of bounded $\mathcal{F}_\alpha$-measurable r.v.'s, i.e. $\int_\alpha^\beta Y H_r dB_r = Y\int_\alpha^\beta H_r dB_r$ if $Y \in L^\infty\left(P, \mathcal{F}_\alpha\right)$.*

Exercise 50. *(see Baldi, 6.9) If $X \in M^2[\alpha,\beta]$ is a continuous process such that $\sup_{\alpha\leq r\leq\beta} X_r \in L^2\left(\mathbb{P}\right)$ then, for every sequence $\left(\pi_n\right)_n$ of partitions $\alpha = t_{n,0} < \ldots < t_{n,m_n} = \beta$, with $|\pi_n| \to 0$, it holds*

$$\lim_{n\to\infty} \sum_{k=0}^{m_n-1} X_{t_{n,k}}\left(B_{t_{n,k+1}} - B_{t_{n,k}}\right) = \int_\alpha^\beta X_r dB_r$$

where the limit is taken in $L^2\left(\mathbb{P}\right)$.

Exercise 51. *Assuming that H has adapted C^1 trajectories on $[\alpha,\beta]$, prove that a.s.*

$$\int_\alpha^\beta H_s dB_s = H_\beta B_\beta - H_\alpha B_\alpha - \int_\alpha^\beta H'_s B_s ds.$$

Exercise 52. *On a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, let X be a real random variable, independent of a σ-algebra $\mathcal{F} \subseteq \mathcal{A}$. Assuming that the law of X is $\mathscr{N}(0,\sigma^2)$, and that Y is a $\mathcal{F}$-measurable real r.v. with $|Y| = 1$ a.s., prove that XY has the same law of X and it is independent of $\mathcal{F}$.*

Exercise 53. *Assume that $(H_s)_{s\geq 0}$ is adapted and $|H_s| = 1$ for any s. Prove that the process $Z_s = \int_0^s H_r dB_r$ is a Brownian motion.*

Exercise 54. *(Wiener integrals) Let $f_1, \ldots, f_n \in L^2((0,T),\mathscr{L})$. Prove that the random variables*

$$F_i = \int_0^T f_i\left(s\right) dB_s \in L^2(\Omega,\mathbb{P}) \quad (i = 1\ldots,n)$$

have joint Gaussian law: compute its mean vector and its covariance matrix.

Let f be a locally bounded Borel function defined on $[0,\infty)$. Prove that the process

$$Z_\cdot = \int_0^\cdot f(s)\, dB_s$$

is Gaussian with independent increments.

Exercise 55. *(Multiple Wiener integrals) Fix $T > 0$, let*

$$\Delta^2 = \left\{(s,t) \in \mathbb{R}^2 | 0 \le s \le t \le T\right\}$$

and call rectangle any set $R =]a,\alpha] \times]b,\beta] \subseteq \Delta^2$. Prove the following statements.

1. *For any rectangle $R =]a,\alpha] \times]b,\beta]$,*

$$t \mapsto \int_0^t I_R(s,t)\, dB_s = \begin{cases} B_\alpha - B_a & \text{if } t \in]b,\beta] \\ 0 & \text{otherwise} \end{cases}$$

   defines a process $I_R \cdot B \in M^2[0,T]$.

2. *For R, R' rectangles, $\langle I_R \cdot B, I_R' \cdot B\rangle_{M^2} = \mathscr{L}^2(R \cap R')$.*

3. *Given a function $f = \sum_{i=1}^n \alpha_i I_{R_i}$ where $\alpha_i \in \mathbb{R}$ and R_i are rectangles,*

$$f \cdot B = \sum_{i=1}^n \alpha_i (I_{R_i} \cdot B) \in M^2[0,T]$$

   does not depend on the representation of f.

4. *The following isometry holds:*

$$\langle f \cdot B, g \cdot B\rangle_{M^2} = \int_{\Delta^2} f(s,t)\, g(s,t)\, ds dt.$$

5. *The map $f \mapsto f \cdot B$ extends to an isometry of $L^2(\Delta^2, \mathscr{L}^2)$ into $M^2[0,T]$, and in particular $f \cdot B$ is well-defined for any $f \in L^2(\Delta^2, \mathscr{L}^2)$ as follows:*

$$\text{given } f_n \to f \text{ in } L^2(\Delta^2, \mathscr{L}^2), \text{ put } f \cdot B = \lim_n (f_n \cdot B).$$

6. *Define the multiple Wiener integral*

$$I_2(f) = \int_0^T dB_t \int_0^t f(s,t)\, dB_s = \int_0^T (f \cdot B)_t\, dB_t \in L^2(\mathbb{P})$$

   and prove that

$$\mathbb{E}\left[\left|\int_0^T dB_t \int_0^t f(s,t)\, dB_s\right|^2\right] = \int_{\Delta^2} f(s,t)\, g(s,t)\, ds dt.$$

Exercise 56. *With the notation of the above exercise, is the process* $(0 \leq r \leq T)$

$$r \mapsto \int_0^r dB_t \int_0^t f(s,t)\, dB_r \in L^2(P)$$

adapted? is it a martingale?

Exercise 57. *Repeat the construction of the exercise above for any* $n > 2$ *and provide a precise definition of*

$$I_n(f) = \int_0^T dB_{t_n} \int_0^{t_n} dB_{t_{n-1}} \int_0^{t_{n-2}} \cdots \int_0^{t_2} dB_{t_1} f(t_1, t_2, \ldots, t_n) \in L^2(\mathbb{P}),$$

for any $f \in L^2(\Delta^n, \mathscr{L}^n)$, *where*

$$\Delta^n = \{(t_1, \ldots, t_n) \in \mathbb{R}^n | 0 \leq t_1 \leq \ldots \leq t_n \leq T\}.$$

Exercise 58. *With the notation of the above exercises, prove the following orthogonality relations, for multiple Wiener integrals: given* $1 \leq n, m \leq m$, $f \in L^2\left(\Delta^n, \mathscr{L}^2\right)$, $g \in L^2\left(\Delta^m, \mathscr{L}^2\right)$, *it holds*

$$\mathbb{E}[I_n(f) I_m(g)] = \begin{cases} \langle f, g\rangle_{L^2(\Delta^n)} & \textit{if } n{=}m \\ 0 & \textit{otherwise.} \end{cases}$$

Moreover, $\mathbb{E}[I_n(f)] = 0$.

11 Itô's Formula

As in the previous section, we assume that we are given $(B_t)_{t\geq 0}$, a real continuous Brownian motion, starting from 0, defined on a filtered probability space $(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{F} = (\mathcal{F}_t)_{t\geq 0})$, where $\mathcal{F}$ satisfies the usual assumptions.

Exercise 59. *Using Itô's formula, find alternative expressions for*

1. $\int_0^t B_s dB_s$, $\int_0^t B_s ds$;
2. B_t^2, $\int_0^t B_s^2 dB_s$;
3. $\cos(B_t)$, $\int_0^t \sin(B_s)\, dB_s$;
4. $\int_0^t \exp\{B_s\}\, dB_s$.

Exercise 60 (Stochastic exponential)**.** *(see Baldi, sec. 7.2) For* $H \in \Lambda^2([0,\infty[)$, *put* $X_t = \int_0^t H_s dB_s$ *and prove that*

$$t \mapsto \mathcal{E}(X)_t = \exp\left\{X_t - \frac{1}{2}\int_0^t |H_s|^2\, ds\right\}$$

is a positive local martingale.

Exercise 61. *Prove that for any* $t > 0$, $\mathbb{E}\left[\mathcal{E}(B)_t\right] = 1$, *and* $\lim_{t\to\infty} \mathcal{E}(B)_t = 0$ *a.s.*

Exercise 62. *Define for* $t \in [0,1[$

$$Z_t = \frac{1}{\sqrt{1-t}} \exp\left\{-\frac{B_t^2}{2(1-t)}\right\}.$$

Show that $t \mapsto Z_t$ *is a positive martingale for* $t \in [0,1[$, *with* $\mathbb{E}[Z_t] = 1$ *and* $\lim_{t\to 1} Z_t = 0$.

Exercise 63 (Exponential inequality)**.** *(see Baldi, prop. 7.5) Let* $H \in \Lambda^2([0,T])$ *and fix* $k > 0$. *Then*

$$\mathbb{P}\left(\sup_{0\le t\le T}\left|\int_0^t H_s dB_s\right| > c,\ \int_0^T H_s^2 ds \le kT\right) \le 2\exp\left\{-\frac{c^2}{2kT}\right\}.$$

Exercise 64. *Prove that, given* $p \ge 2$, $X_t = \int_0^t H_s dB_s$, *for* $H \in \Lambda^2([0,T])$, *it holds*

$$|X_t|^p = \int_0^t p\,|X_s|^{p-1}\, sgn(X_s)\, H_s dB_s + \frac{1}{2}\int_0^t p(p-1)\,|X_s|^{p-2} H_s^2 ds.$$

Exercise 65. *(A first step towards local times) Given* $\epsilon > 0$, *set* $f_\epsilon(x) = \left(x^2 + \epsilon^2\right)^{1/2}$, *use Itô's formula to show that for* $H \in M^2([0,T])$, *it holds, for* $X_t = \int_0^t H_s dB_s$,

$$\mathbb{E}\left[\left(X_T + \epsilon^2\right)^{1/2}\right] = \mathbb{E}\left[\int_0^T \frac{\epsilon^2}{2\left(X_t^2 + \epsilon^2\right)^{3/2}} H_t^2 dt\right].$$

Deduce that, for some absolute constant $C > 0$, *it holds*

$$\limsup_{\epsilon\to 0} \mathbb{E}\left[\int_0^T \frac{1}{2\epsilon} I_{\{|X_t|\le\epsilon\}} H_t^2 dt\right] \le C\mathbb{E}\left[|X_T|\right].$$

Exercise 66 (Burkholder-Davis-Gundy inequalities)**.** *(see Baldi, prop. 7.9) Fix* $p \ge 2$ *and show that there exists some constant* $C_p > 0$ *such that, for any* $T > 0$ *and any* $H \in M^2([0,T])$, *it holds*

$$\mathbb{E}\left(\sup_{0\le t\le T}\left|\int_0^t H_s dB_s\right|^p\right) \le C_p \mathbb{E}\left(\left|\int_0^T H_s^2 ds\right|^{p/2}\right).$$

In particular, this entails that

$$\mathbb{E}\left(\sup_{0\le t\le T}\left|\int_0^t H_s dB_s\right|^p\right) \le C_p T^{\frac{p-2}{2}} \mathbb{E}\left(\int_0^T |H_s|^p ds\right).$$

(Actually, in the first inequality above, the opposite inequality holds too, with a different constant.)

Exercise 67. *Given $H \in \Lambda^2[0,T]$, define recursively $I_n \in \Lambda^2[0,T]$ as follows:*

$$\begin{cases} I_0(t) = 1 \\ I_n(t) = \int_0^t I_{n-1}(s) H_s dB_s \quad \text{for } n \geq 1. \end{cases}$$

Prove that for $n \geq 2$, it holds

$$nI_n(t) = I_{n-1}(t) \int_0^t H_s dB_s - I_{n-2}(t) \int_0^t H_s^2 ds.$$

11.1 Multidimensional Itô's Formula

In what follows, we assume that $n \geq 1$ is fixed we are given $(B_t)_{t \geq 0}$, an n-dimensional Brownian motion, starting from the origin, defined on $(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{F} = (\mathcal{F}_t)_{t \geq 0})$, where $\mathcal{F}$ satisfies the usual assumptions.

Exercise 68. *Let $f \in C_b^2(\mathbb{R}^n)$, i.e. $|\nabla f|$ and $|D^2 f|$ uniformly bounded. Prove that*

1. *if $\Delta f = 0$, then $t \mapsto f(B_t)$ is a martingale;*
2. *if $t \mapsto f(B_t)$ is a martingale, then $\Delta f = 0$.*

Exercise 69. *Let $n = 2$ and define*

$$(X_t, Y_t) = \left(\exp\left(B_t^1\right) \cos\left(B_t^2\right), \exp\left(B_t^1\right) \sin\left(B_t^2\right)\right).$$

Compute the stochastic differential of the process $(X_t - 1)^2 + Y_t^2$.

Exercise 70. *Let $R > 0$, $x_0 \in \mathbb{R}^n$, with $|x_0| < R$ and let $\tau_R = \inf\{t \geq 0 \,:\, |x_0 + B_t| = R\}$. Prove that*

$$\mathbb{E}[\tau_R] = \left(R^2 - |x_0|^2\right)/n.$$

Exercise 71 (Exit time from a spherical shell)**.** *Fix $0 < r < R$, $T > 0$ and $x_0 \in \mathbb{R}^n$, with $r < |x_0| < R$. Let*

$$\tau_r = \inf\{t \geq 0 \,:\, |x_0 + B_t| = r\} \quad \tau_R = \inf\{t \geq 0 \,:\, |x_0 + B_t| = R\} \quad \tau = \tau_r \wedge \tau_R.$$

Assume $n \geq 3$, write down and fully justify Itô formula for $f(x_0 + B_t^\tau)$, where

$$f(x) = |x|^{2-n}.$$

Taking expectations and letting $t \to \infty$, deduce that

$$\mathbb{P}(\tau_r < \tau_R) = \left(|x_0|^{2-n} - R^{2-n}\right) / \left(|r|^{2-n} - R^{2-n}\right).$$

Letting $R \to \infty$, prove that $\mathbb{P}(\tau_r < \infty) = (r/|x_0|)^{n-2} < 1$. Deduce the following assertions.

1. *Points are polar for B_t, i.e.*

$$\mathbb{P}\left(\exists t \geq 0 \,:\, B_t = -x_0\right) = 0.$$

2. *The process leaves every compact set with strictly positive probability, but infact it holds $\lim_{t\to\infty} |B_t(\omega)| = \infty$ a.e. ω. (Hint: strong-Markov and Borel-Cantelli).*

Adapt the steps above for the case $n = 2$, using $f(x) = -\log|x|$ and discuss the validity of the correspondent conclusions.

Exercise 72. *Let $X^1, \dots, X^d$ be Itô processes (defined on the same space, fixed above). Using the* differential *notation $dX^i, d[X_i, X_j]$, write as Itô processes:*

1. $\Pi_{i=1}^n X^i$;
2. X^1/X^2 *(assume $|X^2| > \epsilon$ for some $\epsilon > 0$);*
3. $|X^1|^k$, *where $k \geq 2$,*
4. $\log(X^1)$, *(assume $X^1 > \epsilon$ for some $\epsilon > 0$);*
5. $(X^1)^{X_2}$, *(assume $X^1 > \epsilon$ for some $\epsilon > 0$);*
6. $\sin(X^1)\cos(X^2) + \cos(X^1)\sin(X^2)$;

11.2 Lévy theorem

Exercise 73. *Given an n-dimensional Wiener process B, let $H_s \in \Lambda^2_B([0,\infty])$ with $H_s H_s^* = Id$ for a.e. ω, for all $s \in [0,\infty[$. Prove that*

$$t \mapsto \int_0^t H_s dB_s$$

is a Brownian motion.

Exercise 74. *Given a real valued Wiener process B, let $H_s \in \Lambda^2_B([0,\infty])$ and assume that $\int_0^t H_s^2 ds = f(t)$ is a deterministic process. Prove that*

$$t \mapsto \int_0^t H_s ds$$

is a Gaussian process.

Exercise 75 (Reflection principle)**.** *Let B be a real valued Wiener process, and let $a > 0$. Write $T_a = \inf\{t \geq 0 : B_t \geq a\}$. Define*

$$W_t = \int_0^t \left(I_{\{s<T_a\}} - I_{\{s\geq T_a\}}\right) dB_s$$

and prove that W is a Wiener process. Use this fact and the identity

$$\{T_a < t\,,\, B_t < a\} = \{W_t > a\}$$

to conclude that

$$\mathbb{P}(T_a < t) = \mathbb{P}(B_t > a) + \mathbb{P}(W_t > a) = \mathbb{P}(|B_t| > a)\,.$$

12 SDEs

Exercise 76. *Consider the matrix*

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and the two-dimensional SDE, with additive noise,

$$dX_t = AX_t dt + \sigma dB_t,$$

with $X_0 = (p_0, q_0)$, $\sigma = (0, \sigma_0)$ *and* dB_t *is the stochastic differential of a real (1-dimensional) Brownian motion. Recall the existence and uniqueness results for such an equation and prove that*

$$\mathbb{E}\left[|X_t|^2\right] = |X_0|^2 + \sigma_0^2 t.$$

Exercise 77 (Ornstein-Uhlenbeck process)**.** *Consider a real-valued Wiener process* B *and two real numbers* θ, σ*. Consider the following SDE*

$$dX_t = -\theta X_t dt + \sigma dB_t$$

with $X_0 = x_0$ *and prove the following assertions.*

1. *It holds*
$$X_t = e^{-\theta t} x_0 + e^{-\theta t} \int_0^t e^{\theta s} \sigma dB_s.$$

2. *The process* X_t *is Gaussian.*

3. *If* $\theta > 0$*, then* $\lim_{t \to \infty} X_t$ *exists in in law but not in* $L^2(\Omega, \mathbb{P})$*. Compute its mean and covariance.*

Exercise 78 (Variation of constants formula)**.** *Consider a real-valued Wiener process* B *and two real numbers* θ, σ *and a continuous function* $f : [0, \infty[\times \mathbb{R} \to \mathbb{R}$*. Consider the following SDE,*

$$dX_t = (-\theta X_t + f(t, X_t))\, dt + \sigma dB_t,$$

with $X_0 = x_0$*. Prove the following assertions.*

1. X_t *admits the representation*
$$X_t = e^{-\theta t} x_0 + e^{-\theta t} \int_0^t e^{\theta s} f(X_s)\, ds + e^{-\theta t} \int_0^t e^{\theta s} \sigma dB_s.$$

2. *Assume that* $f(t, x) = g(t)$ *does not depend on* x *and prove that* X_t *is Gaussian.*

3. *Assuming* $f(t, x) = g(t)$ *and that* $\lim_{t \to \infty} e^{-\theta t} \int_0^t e^{\theta s} g(s)\, ds = \mu$*, prove that* $\lim_{t \to \infty} X_t$ *exists in law. Compute its mean and covariance.*

Exercise 79. *Let μ, σ be real numbers, with $\sigma \neq 0$ and let B be a real-valued Wiener process, starting from 0. Consider the following SDE:*

$$dS_t = S_t \left(\mu dt + \sigma dB_t \right), \quad S_0 = s_0 \in]0, \infty[.$$

Prove the following assertions.

1. *Strong existence and uniqueness hold for S_t, which admits the following expression*

$$S_t = s_0 \exp \left\{ \left(\mu - \sigma^2/2 \right) t + \sigma B_t \right\} > 0.$$

2. *The law of $\log S_t$ is Gaussian, i.e. the law of S_t is* log-normal. *Compute its mean and variance.*

3. *Discuss the behaviour of S_t as $t \to \infty$.*

Exercise 80. *Fix $T, K > 0$ and let μ, σ B and S as in the exercise above. Write $\Phi(x) = \int_{-\infty}^{x} exp \left(-t^2/2 \right) / \sqrt{2\pi}$ for the repartition function of a normal distribution.*

1. *Show the following* Black-Scholes formula *for the price at time 0 of a European call option with expiration time T and strike price K:*

$$C = e^{-\mu T} \mathbb{E} \left[\left(S_T - K \right)^+ \right] = s_0 \Phi \left(-c + \sigma \sqrt{T} \right) - K e^{-\mu T} \Phi(-c),$$

where

$$c = \frac{\log \left(K / c_0 \right) - \left(\mu - \sigma^2/2 \right) T}{\sigma \sqrt{T}}.$$

2. *Show* Black-Scholes formula *for the correspondent European put option:*

$$P = e^{-\mu T} \mathbb{E} \left[\left(K - S_T \right)^+ \right] = s_0 \Phi \left(-p + \sigma \sqrt{T} \right) - K e^{-\mu T} \Phi(-p),$$

where

$$p = \frac{\log \left(K / c_0 \right) - \left(\mu + \sigma^2/2 \right) T}{\sigma \sqrt{T}}.$$

3. *Deduce the Put-Call parity for the European option:*

$$C - P = S_0 - K.$$

Exercise 81. *Let $\mu, \mu', \sigma, \sigma'$ be real numbers, and let B be a real-valued Wiener process, starting from 0. Consider the following SDE's:*

$$dS_t = S_t \left(\mu dt + \sigma dB_t \right), \quad S_0 = s_0 \in]0, \infty[,$$

$$dS'_t = S'_t \left(\mu' dt + \sigma' dB_t \right), \quad S_0 = s'_0 \in]0, \infty[$$

where s_0, s'_0 are real numbers, with $s_0 \neq 0$. Prove that if, for some $0 \leq t_1 < t_2$, it holds

$$S_{t_1} = S'_{t_1} \quad S_{t_2} = S'_{t_2}$$

then $s_0 = s'_0$, $\mu = \mu'$ and $\sigma = \sigma'$.

13 Itô processes and stopping times

Exercise 82. *On a filtered probability space* $\left(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{F} = (\mathcal{F}_t)_{t\geq 0}\right)$, *with the usual assumptions, let* τ *be a stopping time, i.e. a* $[0,\infty]$*-valued r.v. such that, for every* $t \geq 0$, $\{\tau \leq t\} \in \mathcal{F}_t$. *Prove that there exists a decreasing sequence of stopping times* τ_n, *converging uniformly to* τ *on* $\{\tau < \infty\}$, *such that* $\tau_n(\omega) \in \mathbb{N}/2^n \cup \{\infty\}$.

Exercise 83. *Let* $H \in M_B^2[0,T]$ *and let* τ *be a stopping time (with respect to the natural filtration of a Brownian motion* B*). Prove that*

$$s \mapsto H_s I_{[0,\tau[}(s)$$

is a process in $M_B^2[0,T]$ *and that, if we write* $X_t = \int_0^t H_s dB_s$, *then*

$$\int_0^T H_s I_{[0,\tau[}(s)\, dB_s = X_{T\wedge\tau}.$$

(Hint: assume first that the range of τ *is discrete and use the locality of the stochastic integral, then use the exercise above)*