Stochastic Partial Differential Equations III

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Plan of the lectures

- The method of compactness
- Kolmogorov equations
- Open problems of regularization by noise

Kolmogorov equations in Hilbert spaces

Consider a separable Hilbert space H, a linear operator $A : D(A) \subset H \to H$, a bounded Borel measurable operator $B : H \to H$, and a selfadjoint bounded operator $Q : H \to H$. We investigate the equation

$$
\begin{aligned}
\partial_t U(t,x) &= \frac{1}{2} Tr\left(QD^2 U(t,x)\right) + \langle Ax + B(x), DU(t,x)\rangle \\
U(0,x) &= U_0(x).
\end{aligned}
$$

The backward form of this equation corresponds to the SDE in the Hilbert space H

$$
dX_t = (AX_t + B(X_t))\, dt + \sqrt{Q} dW_t
$$

which may be the abstract formulation of an SPDE.

The main idea (Da Prato) to investigate

$$\begin{aligned} \partial_t U(t,x) &= \frac{1}{2} Tr\left(QD^2 U(t,x)\right) + \langle Ax + B(x), DU(t,x)\rangle \\ U(0,x) &= U_0(x). \end{aligned}$$

is to solve the "linear" case (Ornstein-Uhlenbeck)

$$\begin{aligned} \partial_t V(t,x) &= \frac{1}{2} Tr\left(QD^2 V(t,x)\right) + \langle Ax, DU(t,x)\rangle \\ V(0,x) &= V_0(x) \end{aligned}$$

using Probability (Gaussian stochastic analysis) and the nonlinear case by perturbation, analytically.

Called $(S_t V_0)(x)$ the solution of

$$\begin{aligned} \partial_t V(t,x) &= \frac{1}{2} Tr\left(QD^2 V(t,x)\right) + \langle Ax, DU(t,x)\rangle \\ V(0,x) &= V_0(x) \end{aligned}$$

we write the original equation in the perturbative form

$$U(t,x) = (S_t U_0)(x) + \int_0^t \left(S_{t-s}\langle B(\cdot), DU(s,\cdot)\rangle\right)(x)\, ds.$$

Linear case

The "linear" case (Ornstein-Uhlenbeck)

$$\begin{aligned} \partial_t V(t,x) &= \frac{1}{2} Tr\left(QD^2 V(t,x)\right) + \langle Ax, DU(t,x)\rangle \\ V(0,x) &= V_0(x) \end{aligned}$$

can be solved explicitly. We do not enter the nontrivial question of the meaning of the terms and in which sense it is satisfied, but simply state, by analogy with the finite dimensional case, that the solution to this equation is given by

$$V(t,x) = \mathbb{E}\left[V_0\left(X_t^x\right)\right]$$

where

$$dX_t^x = AX_t^x dt + \sqrt{Q} dW_t, \qquad X_0^x = x$$

or explicitly (we use semigroup theory here)

$$X_t^x = e^{tA}x + \int_0^t e^{(t-s)A}\sqrt{Q}dW_s.$$

Consider for simplicity the diagonal case, $A = A^*$ with compact resolvent, $Ae_k = -\lambda_k e_k$:

$$X_t^x = \sum_k \left(e^{-t\lambda_k} x_k + \int_0^t e^{-(t-s)\lambda_k} \sigma_k d\beta_s \right) e_k.$$

It is well defined when

$$\sum_k \int_0^T e^{-2(t-s)\lambda_k} \sigma_k^2 ds < \infty$$

namely

$$\sum_k \frac{1 - e^{-2T\lambda_k}}{2\lambda_k} \sigma_k^2 < \infty$$

namely when

$$\sum_k \frac{\sigma_k^2}{\lambda_k} < \infty.$$

Example

On the torus $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ take $H = L^2 \left(\mathbb{T}^d\right)$ with zero mean (scalar valued, for simplicity; heat equation instead of Navier-Stokes), $A = \Delta$,

$$e_k(x) = \exp(2\pi i k \cdot x), \qquad k \in \mathbb{Z}^d$$

$$Ae_k = -4\pi^2 |k|^2 e_k \qquad (\text{namely } \lambda_k = 4\pi^2 |k|^2).$$

If we want to deal with space-time white noise, namely $\sigma_k^2 = 1$, we need

$$\sum_k \frac{\sigma_k^2}{\lambda_k} = \frac{1}{4\pi^2} \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \frac{1}{|k|^2} < \infty$$

hence it is true only in

$$d = 1.$$

Summary of notations, equations, assumptions:

$$U(t,x) = (S_t U_0)(x) + \int_0^t \left(S_{t-s} \langle B(\cdot), DU(s,\cdot)\rangle\right)(x)\, ds$$

$$(S_t V_0)(x) = \mathbb{E}\left[V_0\left(X_t^x\right)\right]$$

$$X_t^x = \sum_k \left(e^{-t\lambda_k} x_k + \int_0^t e^{-(t-s)\lambda_k} \sigma_k d\beta_s \right) e_k$$

$$\sum_k \frac{\sigma_k^2}{\lambda_k} < \infty, \qquad B \in \mathcal{B}_b(H,H).$$

Clearly the difficulty in using iteratively the equation

$$U(t,x) = (S_t U_0)(x) + \int_0^t (S_{t-s} \langle B(\cdot), DU(s,\cdot)\rangle)(x)\, ds$$

is in the derivative $DU(s,\cdot)$ on the RHS. The problem can be solved if we have an estimate of the form

$$\|DS_t V_0\| \leq C(t) \|V_0\|$$

where $\|\cdot\|$ is some kind of uniform or L^p norm.

For general problems, Malliavin calculus or Bismut-Elworthy-Li formula are good tools. Here the process behind S_t is Gaussian and thus we have an explicit formula for $DS_t V_0$ which does not involve derivatives of V_0.

Introduce

$$Q(t) = \int_0^t e^{sA} Q e^{sA^*} ds.$$

It is the covariance operator, in the "space variable", of the process

$$X_t^x = e^{tA}x + \int_0^t e^{(t-s)A} \sqrt{Q} dW_s$$

$$\mathbb{E}\left[\left\langle X_t^x - e^{tA}x, h\right\rangle \left\langle X_t^x - e^{tA}x, k\right\rangle\right] = \langle Q(t) h, k\rangle .$$

If $Q(t)$ is injective, we deduce

$$\mathbb{E}\left[\left\langle X_t^x - e^{tA}x, Q(t)^{-1/2} f\right\rangle \left\langle X_t^x - e^{tA}x, Q(t)^{-1/2} g\right\rangle\right] = \langle f, g\rangle .$$

The following fact is true: when $Q(t)$ is injective, there is a rigorous definition, as an L^2-limit, for the random variable formally denoted by

$$\left\langle Q(t)^{-1/2} \left(X_t^x - e^{tA}x\right), h\right\rangle$$

and it is centered Gaussian:

$$N\left(0, \|h\|_H^2\right).$$

Theorem

Assume $\mathcal{R}\left(e^{tA}\right) \subset \mathcal{R}\left(Q(t)^{1/2}\right)$ *for* $t > 0$*, and set* $\Lambda(t) = Q(t)^{-1/2} e^{tA}$*. Then*

$$\langle h, D(S_t V_0)(x)\rangle = -\mathbb{E}\left[V_0(X_t^x)\left\langle \Lambda(t) h, Q(t)^{-1/2}\left(X_t^x - e^{tA}x\right)\right\rangle\right].$$

Proof.

$$\left(S_t^{(n)} V_0\right)(x) = \mathbb{E}\left[V_0\left(X_t^{x,n}\right)\right]$$

$$\begin{aligned} X_t^x &= \sum_{k=1}^{n}\left(e^{-t\lambda_k}x_k + \int_0^t e^{-(t-s)\lambda_k}\sigma_k d\beta_s\right)e_k \\ &\sim \otimes N\left(e^{-t\lambda_k}x_k, \int_0^t e^{-2(t-s)\lambda_k}\sigma_k^2 ds\right) = \otimes N\left(m_k(t, x_k), \sigma_k^2(t)\right) \end{aligned}$$

With the notation $y = (y_1, ..., y_n)$,

$$\left(S_t^{(n)} V_0\right)(x)$$

$$= \int_{\mathbb{R}^n} V_0(y)\, C(n,t) \exp\left(-\frac{1}{2}\sum_{k=1}^{n} \frac{(y_k - m_k(t,x_k))^2}{\sigma_k^2(t)}\right) dy$$

$$\partial_i \left(S_t^{(n)} V_0\right)(x)$$

$$= -\int_{\mathbb{R}^n} \frac{(y_i - m_i(t,x_k))\, e^{-t\lambda_i}}{\sigma_i^2(t)} V_0(y)\, C(n,t)$$

$$\exp\left(-\frac{1}{2}\sum_{k=1}^{n} \frac{(y_k - m_k(t,x_k))^2}{\sigma_k^2(t)}\right) dy$$

$$= -\mathbb{E}\left[V_0\left(X_t^{x,n}\right) \frac{\left(\left(X_t^{x,n}\right)_i - m_i(t,x_k)\right) e^{-t\lambda_i}}{\sigma_i^2(t)}\right]$$

Hence

$$
\begin{aligned}
& \partial_i \left(S_t^{(n)} V_0 \right) (x) \\
= \; & - \int_{\mathbb{R}^n} \frac{\left(y_i - m_i \left(t, x_k \right) \right) e^{-t\lambda_i}}{\sigma_i^2 \left(t \right)} V_0 \left(y \right) C \left(n, t \right) \\
& \exp \left(-\frac{1}{2} \sum_{k=1}^{n} \frac{\left(y_k - m_k \left(t, x_k \right) \right)^2}{\sigma_k^2 \left(t \right)} \right) dy \\
= \; & -\mathbb{E} \left[V_0 \left(X_t^{x,n} \right) \frac{\left(\left(X_t^{x,n} \right)_i - m_i \left(t, x_k \right) \right) e^{-t\lambda_i}}{\sigma_i^2 \left(t \right)} \right]
\end{aligned}
$$

Hence

$$\begin{aligned}\langle h, D(S_t V_0)(x)\rangle &= -\mathbb{E}\left[V_0(X_t^x)\sum_{i=1}^{\infty}\frac{((X_t^x)_i - m_i(t,x_k))\,e^{-t\lambda_i}}{\sigma_i^2(t)}h_i\right] \\ &= -\mathbb{E}\left[V_0(X_t^x)\left\langle Q(t)^{-1}e^{tA}h, X_t^x - e^{tA}x\right\rangle\right].\end{aligned}$$

Now the question is how to use the formula

$$\langle h, D(S_t V_0)(x)\rangle = -\mathbb{E}\left[V_0(X_t^x)\left\langle \Lambda(t) h, Q(t)^{-1/2}\left(X_t^x - e^{tA}x\right)\right\rangle\right]$$

We know that

$$\left\langle \Lambda(t) h, Q(t)^{-1/2}\left(X_t^x - e^{tA}x\right)\right\rangle \sim N\left(0, \|\Lambda(t) h\|_H^2\right).$$

Hence we simply have

Theorem

$$|\langle h, D(S_t V_0)(x)\rangle| \leq \|\Lambda(t) h\|_H \, \mathbb{E}\left[|V_0(X_t^x)|^2\right]^{1/2}.$$

In particular,

$$\|D(S_t V_0)(x)\|_H \leq \|\Lambda(t)\|_{\mathcal{L}(H,H)} \|V_0\|_\infty$$

$$\int_H \|D(S_t V_0)(x)\|_H^2 \mu(dx) \leq \|\Lambda(t)\|_{\mathcal{L}(H,H)}^2 \int_H \mathbb{E}\left[|V_0(X_t^x)|^2\right] \mu(dx)$$

The problem is to have a good control of $\|\Lambda(t)\|_{\mathcal{L}(H,H)}$ for $t \to 0$.

The first inequality

$$\|D(S_t V_0)(x)\|_H \leq \|\Lambda(t)\|_{\mathcal{L}(H,H)} \|V_0\|_\infty$$

will be used in the C^0-theory. The second one

$$\int_H \|D(S_t V_0)(x)\|_H^2 \mu(dx) \leq \|\Lambda(t)\|_{\mathcal{L}(H,H)}^2 \int_H \mathbb{E}\left[|V_0(X_t^x)|^2\right] \mu(dx)$$

in the L^2-theory, taking as μ an invariant measure of X_t^x, because in such a case

$$\int_H |V_0(X_t^x)|^2 \mu(dx) = \int_H |V_0(x)|^2 \mu(dx)$$

hence

$$\int_H \|D(S_t V_0)(x)\|_H^2 \mu(dx) \leq \|\Lambda(t)\|_{\mathcal{L}(H,H)}^2 \|V_0\|_{L^2(H,\mu)}^2 .$$

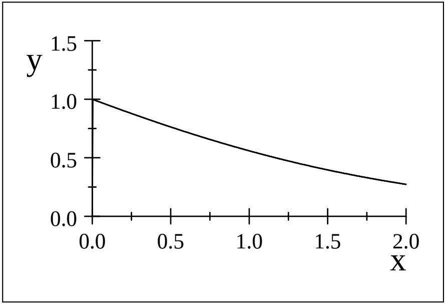
The problem, as said above, is to have a good control of $\|\Lambda(t)\|_{\mathcal{L}(H,H)}$ for $t \to 0$, $\Lambda(t) = Q(t)^{-1/2} e^{tA}$, $Q(t) = \int_0^t e^{sA} Q e^{sA^*} ds$.
If the noise is "regular in the space variable" then $Q(t)^{-1}$ is a very "irregular" operator. Consequence: this approach works for cylindrical noise (space-time white noise) or little modifications but not in general.
Example: $Q = I$, $A = A^*$, $t \in [0, T]$:

$$Q(t) e_k = \int_0^t e^{2sA} e_k ds = \frac{1 - e^{-2t\lambda_k}}{2\lambda_k}$$

$$\Lambda(t) e_k = \frac{\sqrt{2\lambda_k} e^{-t\lambda_k}}{\sqrt{1 - e^{-2t\lambda_k}}} = \frac{1}{\sqrt{t}} g(t\lambda_k), \qquad g(s) \leq 1$$

$$\|\Lambda(t)\|_{\mathcal{L}(H,H)} \leq \frac{1}{\sqrt{t}}.$$

$$g(s) = \frac{\sqrt{2s}e^{-s}}{\sqrt{1-e^{-2s}}}, \qquad g(s) \leq 1$$



The analogous result for second derivatives is similar but more lengthy and we omit the proof.

Theorem

$$\begin{aligned}
& \left\langle k, D^2 \left(S_t V_0 \right) (x)\, h \right\rangle \\
= \; & -\mathbb{E}\left[V_0 \left(X_t^x \right) \cdot \left\{ \left\langle \Lambda(t)\, h, Q(t)^{-1/2} \left(X_t^x - e^{tA} x \right) \right\rangle \cdot \right.\right. \\
& \left.\left. \cdot \left\langle \Lambda(t)\, k, Q(t)^{-1/2} \left(X_t^x - e^{tA} x \right) \right\rangle - \left\langle \Lambda(t)\, h, \Lambda(t)\, k \right\rangle \right\} \right]
\end{aligned}$$

Theorem

$$\left\| D^2 \left(S_t V_0 \right) (x) \right\| \leq \sqrt{2} \left\| \Lambda(t) \right\|^2_{\mathcal{L}(H,H)} \left\| V_0 \right\|_\infty .$$

With these results in our hands we approach

$$U(t,x) = (S_t U_0)(x) + \int_0^t \left(S_{t-s} \langle B(\cdot), DU(s,\cdot) \rangle \right)(x)\, ds$$

in the space

$$U \in C\left([0,T]; C^1(H, \mathbb{R})\right)$$

under the assumption

$$B \in C([0,T]; C(H,H)).$$

Theorem

If $\lim_{t \to 0} \int_0^t \|\Lambda(s)\|_{\mathcal{L}(H,H)}\, ds = 0$, given $U_0 \in C^1(H, \mathbb{R})$ there is a unique solution $U \in C\left([0,T]; C^1(H,\mathbb{R})\right)$ of the Kolmogorov equation.

The idea of proof is simply a fixed point in the space $C\left([0,T];C^1(H,\mathbb{R})\right)$. Contraction of the map

$$\Lambda : C\left([0,T];C^1(H,\mathbb{R})\right) \to C\left([0,T];C^1(H,\mathbb{R})\right)$$

$$\Lambda(U)(t) = (S_t U_0)(x) + \int_0^t (S_{t-s}\langle B(\cdot), DU(s,\cdot)\rangle)(x)\, ds$$

is guaranteed by the estimate

$$\begin{aligned} & \|D\Lambda(U_1)(t) - D\Lambda(U_2)(t)\|_\infty \\ \leq & \int_0^t \|DS_{t-s}\langle B(\cdot), DU_1(s,\cdot) - DU_2(s,\cdot)\rangle\|_\infty\, ds \\ \leq & \int_0^t \|\Lambda(t-s)\|_{\mathcal{L}(H,H)} \|\langle B(\cdot), DU_1(s,\cdot) - DU_2(s,\cdot)\rangle\|_\infty\, ds \end{aligned}$$

where we have used

$$\|D(S_t V_0)(x)\|_H \leq \|\Lambda(t)\|_{\mathcal{L}(H,H)} \|V_0\|_\infty .$$

Putting together all the previous pieces plus some additional detail we get:

Theorem

Consider the SDE in Hilbert space H

$$dX_t = (AX_t + B(X_t))\,dt + dW_t, \qquad X_0 = x_0$$

where W_t is cylindrical, $A = A^*$, A^{-1} is trace class, $B : H \to H$ is continuous and bounded. Then there exists a unique solution in law.

With some more effort based on the second derivatives and the assumption

$$B \in C_b^{\alpha}(H, H)$$

it is possible to prove pathwise uniqueness (Da Prato-F. JFA 2010). With more effort one can work in $L^2(H, \mu)$ where μ is the invariant measure of the linear equation and prove a similar result when

$$B \in L^{\infty}(H, H)$$

(Da Prato-F.-Priola-Röckner, AoP 2013). However, the result in this case has been proved only for μ-a.e. $x_0 \in H$.

Open problem 1

The main open problem is to extend previous results to drifts B relevant for Mathematical Physics.
An example like Navier-Stokes has the spaces and operators:

$$\begin{aligned} H &\sim L^2 \text{ (precisely vector valued, divergence free with b.c.)} \\ A &\sim \Delta \text{ (up to projection to divergence free fields)} \\ B(u) &= -u \cdot \nabla u \text{ (as above)} \end{aligned}$$

$$du = (Au + B(u))\, dt + \sqrt{Q} dW_t.$$

The operator B is just "polynomial" in the variables (not irregular as a function only of class $C^\alpha(H, H)$) but:

- quadratically unbounded, opposite to $\|B(u)\|_H \leq C$
- defined only on subspaces ($B : W^{s,2} \to H$ for s large enough), or with range in larger spaces ($B : H \to W^{-r,2}$).

Think to the calculation

$$
\begin{aligned}
& \left\| D\Lambda\left(U_1\right)(t) - D\Lambda\left(U_2\right)(t)\right\|_\infty \\
\leq & \int_0^t \left\| DS_{t-s}\left\langle B\left(\cdot\right), DU_1\left(s,\cdot\right) - DU_2\left(s,\cdot\right)\right\rangle\right\|_\infty ds \\
\leq & \int_0^t \left\|\Lambda\left(t-s\right)\right\|_{\mathcal{L}(H,H)} \left\|\left\langle B\left(\cdot\right), DU_1\left(s,\cdot\right) - DU_2\left(s,\cdot\right)\right\rangle\right\|_\infty ds.
\end{aligned}
$$

When B is not bounded, how to close the estimate?
When B is not defined over H but only on a smaller space, what kind of properties are needed to make estimates?

Da Prato and Debussche JMPA 2003 succeded to investigate precisely the case of 3D Navier-Stokes equations but the difficulty mentioned above reflected into poor estimates on the derivatives of Kolmogorov equations.

These estimates are not sufficient to prove uniqueness.

Thus the question of good results on Kolmogorov equations when B is like those of fluid mechanics (or other applications in Mathematical Physics) remains open.

Open problem 2

Another question is about degenerate Kolmogorov equations. This is motivated by the transport noise used for Euler equations:

$$d\omega + u \cdot \nabla \omega dt + \sum_{n=1}^{\infty} \sigma_n e_n \cdot \nabla \omega \circ d\beta_n = 0.$$

In abstract terms: the diffusion coefficient is state-dependent and vanishes when $\nabla \omega = 0$.
The associated Kolmogorov equation is

$$\partial_t U(t,\omega) + \langle u(\omega) \cdot \nabla \omega, DU(t,\omega) \rangle$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \sigma_n^2 \langle e_n \cdot \nabla \omega, D \langle e_n \cdot \nabla \omega, DU(t,\omega) \rangle \rangle .$$

It is a fully open problem how to study this equation in uniform (Hölder) topologies. We have done some preliminary work in $L^2(H,\mu)$ spaces (F.-Luo, arXiv).

In finite dimensions it would be

$$dX_t = b(X_t) + \sum_{n=1}^{N} C_n X_t \circ dW_n$$

$$\partial_t U = b(x) \cdot \nabla U + \frac{1}{2} \sum_{n=1}^{N} C_n x \cdot \nabla (C_n x \cdot \nabla U).$$

Open problem 3

Finally let us mention an open problem related to applications.
Assume an equation of the form

$$du = (Au + B(u))\, dt + \sqrt{Q} dW_t$$

is used in some application, like weather or climate prediction.
Given an uncertain initial condition, namely a random variable u_0, maybe our theorems guarantee well posedness of the SPDE.
Which is the probability to have, at the future time T, a value of some quantity larger than a thresold?

$$P(\Phi(u(T)) > \lambda) =?$$

A classical method is to simulate several times the SPDE, namely for several realizations of the i.c. and the noise. Very expensive! Monte Carlo has a rate of convergence of order $\frac{1}{\sqrt{N}}$.
In principle, the law of $u(t)$, call it μ_t, satisfies the Fokker-Planck equation, in weak form

$$\int_H F(t,x)\,\mu_t(dx) - \int_H F(0,x)\,\mu_0(dx) = \int_0^t (\partial_t F + \mathcal{L}F)\,\mu_s(dx)\,ds$$

where $\mathcal{L}F$ is the Kolmogorov operator. Can we simulate the Fokker-Planck equation?

$$P(\Phi(u(T)) > \lambda) = \mu_T\{y \in H : \Phi(y) > \lambda\}.$$

Or alternatively, can we simulate the Kolmogorov equation (it gives expected values)?

Since numerical simulation of parabolic PDEs is limited in practice to small dimension (usually ≤ 3), we have no chance to apply a trivial finite dimensional approximation.
New ideas?

- Maybe it is possible to design efficient schemes based on

$$U(t,x) = (S_t U_0)(x) + \int_0^t (S_{t-s} \langle B(\cdot), DU(s,\cdot)\rangle)(x)\, ds.$$

- Spectral methods based on Fourier analysis of $L^2(H,\mu)$? (cf. Delgado-F., IDAQP 2016).

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