Stochastic Partial Differential Equations II

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Plan of the lectures

- The method of compactness
- Kolmogorov equations
- Open problems of regularization by noise

Kolmogorov equations

Let us recall first some elements in finite dimensions.
In $\mathbb{R}^d$, consider the SDE

$$dX_t = b(t, X_t)\, dt + \sigma dW_t, \qquad X_0 = x_0$$

where

$$b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$$

is at least measurable, σ is a real number, $x_0 \in \mathbb{R}^d$. The associated backward Kolmogorov equation is

$$\begin{aligned} \partial_t u(t, x) + \frac{\sigma^2}{2} \Delta u(t, x) + b(t, x) \cdot \nabla u(t, x) &= 0 \\ u(T, x) &= u_0(x). \end{aligned}$$

We may rewirte it forward simply by setting

$$U(t,x) = u(T-t,x), \qquad B(t,x) = b(T-t,x)$$

$$\begin{aligned} \partial_t U(t,x) &= \frac{\sigma^2}{2}\Delta U(t,x) + B(t,x)\cdot\nabla U(t,x) \\ U(0,x) &= u_0(x). \end{aligned}$$

Any time it is confusing to state a theorem for the backward equation, we formulate it for the corresponding forward one. But in doing computations involving both the SDE and Kolmogorov equation, it is better to use the backward formulation.

Assume X_t is a solution of the SDE and $u(t,x)$ is a solution of class $C^{1,2}$ of backward Kolmogorov equation. Then

$$\begin{aligned} du(t,X_t) &= \left(\partial_t u(t,X_t) + \frac{\sigma^2}{2}\Delta u(t,X_t) + b(t,X_t)\cdot\nabla u(t,X_t)\right)dt \\ &\quad +\sigma\nabla u(t,X_t)\cdot dW_t \end{aligned}$$

hence

$$du(t,X_t) = \sigma\nabla u(t,X_t)\cdot dW_t.$$

If

$$\mathbb{E}\int_0^T |\nabla u(t,X_t)|^2\,dt < \infty$$

then

$$\mathbb{E}\left[u_0(X_T)\right] = u(0,x_0).$$

The formula

$$\mathbb{E}\left[u_0\left(X_T\right)\right] = u\left(0, x_0\right)$$

can be generalized to

$$\mathbb{E}\left[\phi\left(X_t\right)\right] = u_{\phi,t}\left(0, x_0\right)$$

by taking as $u_{\phi,t}$ the solution of the backward equation

$$\begin{aligned} \partial_s u_{\phi,t}\left(s,x\right) + \frac{\sigma^2}{2}\Delta u_{\phi,t}\left(s,x\right) + b\left(s,x\right)\cdot\nabla u_{\phi,t}\left(s,x\right) &= 0 \text{ on } [0,t] \\ u_{\phi,t}\left(t,x\right) &= \phi\left(x\right). \end{aligned}$$

If μ_t denotes the law of X_t, the previous formula identifies μ_t when ϕ can be taken arbitrarily:

$$\int_{\mathbb{R}^d} \phi\left(x\right)\mu_t\left(dx\right) = u_{\phi,t}\left(0, x_0\right).$$

Similarly, the formula

$$\mathbb{E}\left[u_0\left(X_T\right)\right] = u\left(0, x_0\right)$$

can be generalized to

$$u\left(t, x\right) = \mathbb{E}\left[u_0\left(X_T^{t,x}\right)\right]$$

by taking as $X_T^{t,x}$ the solution of the SDE

$$d_s X_s^{t,x} = b\left(s, X_s^{t,x}\right) ds + \sigma dW_s \text{ on } \left[t, T\right], \qquad X_t^{t,x} = x.$$

The previous formula gives a probabilistic representation of the solution $u\left(t, x\right)$ of a PDE in terms of solutions of SDE's.

Let us go back to the link $\left(\mu_t = \mathcal{L}\left(X_t\right)\right)$

$$
\begin{aligned}
\mathbb{E}\left[\phi\left(X_t\right)\right] &= u_{\phi,t}\left(0, x_0\right) \\
\int_{\mathbb{R}^d} \phi\left(x\right)\mu_t\left(dx\right) &= u_{\phi,t}\left(0, x_0\right)
\end{aligned}
$$

where

$$
dX_t = b\left(t, X_t\right)dt + \sigma dW_t, \qquad X_0 = x_0
$$

and $u_{\phi,t}$ the solution of the backward equation

$$
\begin{aligned}
\partial_s u_{\phi,t}\left(s,x\right) + \frac{\sigma^2}{2}\Delta u_{\phi,t}\left(s,x\right) + b\left(s,x\right)\cdot\nabla u_{\phi,t}\left(s,x\right) &= 0 \text{ on } \left[0,t\right] \\
u_{\phi,t}\left(t,x\right) &= \phi\left(x\right).
\end{aligned}
$$

It may be used to prove uniqueness in law for the SDE: the law μ_t is identified (then one has to develop a further argument to pass from marginals to the law on path space; there are different methods, see Stroock-Varadhan or Ambrosio-Trevisan).

Conceptually: an existence result for backward Kolmogorov corresponds to a uniqueness result in law for the SDE.

This is not surprising if we recall that μ_t satisfies the forward Kolmogorov equation, more properly called Fokker-Planck equation, and the backward Kolmogorov equation is the formal dual of Fokker-Planck equation.

Hence it is a form of the usual duality principle between existence and uniqueness.

Fokker-Planck equation

If $\phi(t,x)$ is a smooth compact support test function, then

$$d\phi(t,X_t) = \left(\partial_t\phi(t,X_t) + \frac{\sigma^2}{2}\Delta\phi(t,X_t) + b(t,X_t)\cdot\nabla\phi(t,X_t)\right)dt + \sigma\nabla\phi(t,X_t)\cdot dW_t$$

hence (here we have $\mathbb{E}\int_0^T |\nabla\phi(t,X_t)|^2\,dt < \infty$)

$$\mathbb{E}[\phi(T,X_T)] - \phi(0,x_0)$$

$$= \mathbb{E}\int_0^T \left(\partial_t\phi(t,X_t) + \frac{\sigma^2}{2}\Delta\phi(t,X_t) + b(t,X_t)\cdot\nabla\phi(t,X_t)\right)dt$$

$$\langle\mu_T,\phi_T\rangle - \langle\mu_0,\phi_0\rangle = \int_0^T \left\langle \mu_t, \partial_t\phi_t + \frac{\sigma^2}{2}\Delta\phi_t + b\cdot\nabla\phi_t\right\rangle dt.$$

This is the weak form of Fokker-Planck equation

$$\langle \mu_T, \phi_T \rangle - \langle \mu_0, \phi_0 \rangle = \int_0^T \left\langle \mu_t, \partial_t \phi_t + \frac{\sigma^2}{2} \Delta \phi_t + b \cdot \nabla \phi_t \right\rangle dt.$$

If μ_t has a smooth density $\rho(t,x)$, then

$$\partial_t \rho(t,x) = \frac{\sigma^2}{2} \Delta \rho(t,x) - \operatorname{div}(b(t,x)\rho(t,x)).$$

We see that backward Kolmogorov is the formal dual of this equation.

Uniqueness using Kolmogorov

Obviously when b is locally Lipschitz continuous, uniqueness for the SDE can be proved directly, by Gronwall type arguments.

When b is less regular and $\sigma = 0$, uniqueness usually fails, as the classical Peano examples show.

In the stochastic case, $\sigma \neq 0$, uniqueness is "restored" by noise, even if b is considerably less regular than locally Lipschitz.

One of the major tools is the use of Kolmogorov equations (others are Girsanov formula, Davie methods).

It is necessary here to enter the difference between uniqueness in law and pathwise uniqueness.
For the SDE

$$dX_t = b(t, X_t)\, dt + \sigma dW_t \text{ for } t \in [0, T], \qquad X_0 = x_0$$

we say we have uniqueness in law when the law is the same on $C([0, T]; \mathbb{R}^d)$ for all possible filtered probability spaces $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, d-dimensional $\mathcal{F}_t$-BMs W and continuous $\mathcal{F}_t$-adapted processes X solving the SDE on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$.
We say we have pathwise uniqueness when, given an arbitrary probability space $(\Omega, \mathcal{F}, P)$ and d-dimensional BMs W, if $X_t^{(1)}, X_t^{(2)}$ are continuous adapted processes solving the SDE on $(\Omega, \mathcal{F}, P)$, then $X_t^{(1)}, X_t^{(2)}$ are indistinguishable.

I want to explain the rough principle:

- if Kolmogorov equation has solutions with reasonable gradient estimates, then we have uniqueness in law
- if Kolmogorov equation has solutions with uniform gradient estimates and reasonable control on second derivatives, we have pathwise uniqueness.

In both cases, if b is so irregular that there could be examples of non-uniqueness for $\sigma = 0$, we say that there is a regularization by noise.

The first half of the principle is already clear, at informal level: we have seen above that if

$$\mathbb{E}\int_0^t \left|\nabla u_{\phi,t}\left(t,X_t\right)\right|^2 dt < \infty$$

then we prove

$$\int_{\mathbb{R}^d} \phi\left(x\right)\mu_t\left(dx\right) = u_{\phi,t}\left(0,x_0\right)$$

hence μ_t is identified (then one has to work more to prove uniqueness of the law on path space).
[In fact properties of $\nabla u_{\phi,t}$ enter also in making rigorous the application of Itô formula; this is a difficult technical issue.]
This strategy has been developed even to the extreme case when b is a distribution of suitable class. We do not insist too much on this approach however, in finite dimensions, since for most classes of b one can prove also pathwise uniqueness.

The second half of the principle, namely pathwise uniqueness under more informations on Kolmogorov equation, will be now explained in the following particular case:

$$b \in C\left([0,T]; C_b^\alpha\left(\mathbb{R}^d, \mathbb{R}^d\right)\right).$$

The relevant information is that, under this assumption, the vector-valued non-homogeneous Kolmogorov equation

$$\begin{aligned} \partial_t u(t,x) + \frac{\sigma^2}{2}\Delta u(t,x) + b(t,x)\cdot\nabla u(t,x) &= -b(t,x) + \lambda u(t,x) \\ u(T,x) &= 0 \end{aligned}$$

for $\lambda \geq 0$ has a unique solution of class $C^{1,2}$, with bounded first and second derivatives, with the additional information ($\lambda > 0$)

$$\|\nabla u\|_\infty \leq \frac{C}{\sqrt{\lambda}}.$$

We thus have

$$\begin{aligned} du(t,X_t) &= \left(\partial_t u(t,X_t) + \frac{\sigma^2}{2}\Delta u(t,X_t) + b(t,X_t)\cdot\nabla u(t,X_t)\right)dt \\ &\quad +\sigma\nabla u(t,X_t)\cdot dW_t \\ &= (-b(t,X_t)+\lambda u(t,X_t))\,dt + \sigma\nabla u(t,X_t)\cdot dW_t \end{aligned}$$

namely

$$\begin{aligned} \int_0^t b(s,X_s)\,ds &= u(0,X_0) - u(t,X_t) \\ &\quad +\lambda\int_0^t u(s,X_s)\,ds + \int_0^t \sigma\nabla u(s,X_s)\cdot dW_s. \end{aligned}$$

Therefore the integral equation for X_t

$$X_t = x_0 + \int_0^t b(s, X_s)\, ds + \sigma W_t$$

can be rewritten as

$$\begin{aligned} X_t &= x_0 + u(0, x_0) - u(t, X_t) + \lambda \int_0^t u(s, X_s)\, ds \\ &\quad + \int_0^t \sigma(\nabla u(s, X_s) + I) \cdot dW_s. \end{aligned}$$

The advantage is that instead of the non-Lipschitz function b we have now the Lipschits functions u and ∇u.

The integral terms will be dealt with by Gronwall lemma. The non-integral term $u(t, X_t)$ requires the condition

$$\|\nabla u\|_\infty \leq \frac{C}{\sqrt{\lambda}}$$

in order to be contractive.

Assume $X_t^{(1)}, X_t^{(2)}$ are two solutions and $Y_t = X_t^{(1)} - X_t^{(2)}$. Then

$$
\begin{aligned}
Y_t &= u\left(t, X_t^{(2)}\right) - u\left(t, X_t^{(1)}\right) + \lambda \int_0^t \left(u\left(s, X_s^{(1)}\right) - u\left(s, X_s^{(2)}\right)\right) ds \\
&\quad + \int_0^t \sigma \left(\nabla u\left(s, X_s^{(1)}\right) - \nabla u\left(s, X_s^{(2)}\right)\right) \cdot dW_s
\end{aligned}
$$

$$
\mathbb{E}\left[|Y_t|^2\right] \leq \frac{C}{\lambda}\mathbb{E}\left[|Y_t|^2\right] + C\left(\lambda + \sigma^2 \left\|D^2 u\right\|_\infty^2\right) \int_0^t \mathbb{E}\left[|Y_s|^2\right] ds
$$

hence $X^{(1)} = X^{(2)}$.

Stochastic flows

The same proof, applied to two solutions X_t^x, X_t^y starting from initial conditions

$$X_0^x = x, \qquad X_0^y = y$$

provides the estimate

$$\mathbb{E}\left[\left|X_t^x - X_t^y\right|^2\right] \leq C\left|(x + u(0,x)) - (y + u(0,y))\right|^2$$

$$+\frac{C}{\lambda}\mathbb{E}\left[\left|X_t^x - X_t^y\right|^2\right] + C\left(\lambda + \sigma^2\left\|D^2 u\right\|_\infty^2\right)\int_0^t \mathbb{E}\left[\left|X_s^x - X_s^y\right|^2\right] ds$$

which easily implies (for $t \in [0, T]$)

$$\mathbb{E}\left[\left|X_t^x - X_t^y\right|^2\right] \leq C\left|x - y\right|^2.$$

With moderate effort it can be improved to

$$\mathbb{E}\left[\left|X_t^x - X_s^y\right|^p\right] \leq C_p\left(|t-s|^{p/2} + |x-y|^p\right)$$

for every $p \geq 2$ hence, by Kolmogorov regularity criterium, it provides the existence of a continuous version

$$(t,x) \mapsto X_t^x$$

which is also $(\beta, 2\beta)$-Hölder (locally) for every $\beta < 1/2$. This is the stochastic flow associated to the SDE.
With some additional effort at the PDE level ($u \in C\left([0,T]; C^{2,\alpha}\left(\mathbb{R}^d\right)\right)$, because $b \in C\left([0,T]; C_b^{\alpha}\left(\mathbb{R}^d, \mathbb{R}^d\right)\right)$) and at the stochastic level, one can prove that

$$x \mapsto X_t^x$$

is in fact of class $C_{loc}^{1,\alpha'}\left(\mathbb{R}^d, \mathbb{R}^d\right)$ for every $\alpha' < \alpha$.

Thus we have seen that additive noise in a SDE with only α-Hölder drift b "regularizes" the SDE in the following sense: pathwise uniqueness holds and the dependence on initial conditions is pathwise smooth, $C^{1,\alpha'}_{loc}\left(\mathbb{R}^d, \mathbb{R}^d\right)$ for every $\alpha' < \alpha$.
There are extensions to L^∞-drift and also to $L^q\left(0, T; L^p\left(\mathbb{R}^d, \mathbb{R}^d\right)\right)$ drift for $p, q \geq 2$ satisfying

$$\frac{d}{p} + \frac{2}{q} < 1$$

and some result also for $\frac{d}{p} + \frac{2}{q} = 1$ (Ladysenskaya-Prodi-Serrin condition).
Many people contributed to these results: Zvonkin, Veretennikov, Krylov, Röckner, Gubinelli, Priola, Fedrizzi, Maurelli, Beck, Proske, Mohammed, Nilssen and others.

Regularization by noise for transport equation

As a by-product, these results on stochastic flows imply well-posedness results for stochastic transport and continuity equations.
Consider the transport equation

$$\partial_t u(t,x) + b(t,x) \cdot \nabla u(t,x) = 0, \qquad u(0,x) = u_0(x)$$

when

$$b \in C\left([0,T]; C_b^{\alpha}\left(\mathbb{R}^d, \mathbb{R}^d\right)\right).$$

There are examples of non-uniqueness and examples of loss of regularity (blow-up).

Di Perna - P.L. Lions theory tells us that existence and uniqueness of weak L^∞ solutions is true when

$$b \in W^{1,1}_{loc}, \qquad [\mathrm{div}\, b]^- \text{ bounded}$$

plus growth conditions. But Hölder drift is too weak.

Theorem (F.-Gubinelli-Priola, Inv.Math. 2010)

If $b \in C\left([0,T]; C^\alpha_b\left(\mathbb{R}^d, \mathbb{R}^d\right)\right)$, $\mathrm{div}\, b \in L^p$, $\sigma \neq 0$, then the stochastic transport equation

$$du + b \cdot \nabla u dt + \sigma \nabla u \circ dW = 0, \qquad u(0,x) = u_0(x)$$

is well posed in the class of weak L^∞ solutions.

The heuristic idea of proof using stochastic flows is elementary: by Itô-Wentzel formula in Stratonovich form, if u is a solution of

$$du + b \cdot \nabla u dt + \sigma \nabla u \circ dW = 0, \qquad u(0,x) = u_0(x)$$

and

$$(t,x) \mapsto X_t^x$$

is the stochastic flow, we get

$$u(t, X_t^x) = u_0(x)$$

hence $u(t, X_t^x)$ is uniquely identified:

$$\begin{aligned} & du(t, X_t^x) \overset{\text{Strat}}{=} (du)(t, X_t^x) + \nabla u(t, X_t^x) \circ dX_t^x \\ = & -(b \cdot \nabla u dt + \sigma \nabla u \circ dW) + \nabla u \circ (bdt + \sigma dW) \\ = & \ 0 \end{aligned}$$

The rigorous proof requires to apply rigorously Itô-Wentzel formula in Stratonovich form, hence requires regularization of u. But the regularization

$$u_\epsilon(t,x) := (\theta_\varepsilon * u(t))(x)$$

satisfies (in weak form) an equation with a remainder

$$du_\epsilon + b\cdot\nabla u_\epsilon dt + \sigma\nabla u_\epsilon \circ dW = R_\epsilon dt, \qquad u_\epsilon(0,x) = (\theta_\varepsilon * u_0)(x)$$

where R_ϵ is the commutator

$$R_\epsilon = b\cdot\nabla u_\epsilon - \theta_\varepsilon * (b\cdot\nabla u).$$

Thus a very careful commutator estimate is necessary to complete the proof.

Assume ω is a solution of the 2D Euler equation in vorticity form:

$$d\omega + u \cdot \nabla \omega dt + \sigma \nabla \omega \circ dW = 0, \qquad \omega(0,x) = \omega_0(x)$$

and X_t^x is a solution of

$$dX_t^x = u(t, X_t^x) dt + \sigma dW_t.$$

If we may apply Itô-Wentzel formula in Stratonovich form, we get

$$\begin{aligned} d\omega(t, X_t^x) &\overset{\text{Strat}}{=} (d\omega)(t, X_t^x) + \nabla \omega(t, X_t^x) \circ dX_t^x \\ &= -(u \cdot \nabla \omega dt + \sigma \nabla \omega \circ dW) + \nabla \omega \circ (udt + \sigma dW) \\ &= 0 \end{aligned}$$

so

$$\omega(t, X_t^x) = \omega_0(x)$$

and again we identify uniquely $\omega(t, X_t^x)$.

Unfortunately the previous argument contains some deep false step. Indeed:

1. the same argument works for $\sigma = 0$, where uniqueness is an open problem
2. setting $v(t, x) = u(t, x \pm \sigma W_t)$ one can pass from the stochastic to the deterministic case and viceversa, hence any pathology of one case exists also for the other one.

The second objection is removed by taking a space-dependent noise, see below.

The first objection is more hidden: why the computation is false in the deterministic case and where the noise could improve it? Some difficulties are:

- solutions $\omega(t)$ of class L^2 are not smooth enough to apply chain rules and thus we need to regularize and control commutators.
- the drift u is *random* in the equation

$$dX_t^x = u(t, X_t^x)\, dt + \sigma dW_t$$

and only of class $W^{1,2}$. Making it rigorous remains open.

Space-dependent noise

It means (as in the additive noise case) that we consider

$$W_t(x) = \sum_{n=1}^{\infty} \sigma_n \beta_n(t)\, e_n(x)$$

and thus the SPDE is

$$d\omega + u \cdot \nabla \omega dt + \sum_{n=1}^{\infty} \sigma_n e_n \cdot \nabla \omega \circ d\beta_n = 0$$

and the SDE is

$$dX_t^x = u\left(t, X_t^x\right) dt + \sum_{n=1}^{\infty} \sigma_n e_n\left(X_t^x\right) \circ d\beta_n(t).$$

It looks more difficult but it is more promising for "regularization by noise". Indeed we have at least one positive result.
Consider measure-valued vorticities of the form (point vortices)

$$\omega(t) = \sum_{i=1}^{N} a_i \delta_{X_t^i}$$

where (coherently with Euler equation)

$$dX_t^i = \sum_{j \neq i} \frac{a_j}{2\pi} \frac{\left(X_t^i - X_t^j\right)^{\perp}}{\left|X_t^i - X_t^j\right|^2} dt + \sum_{n=1}^{\infty} \sigma_n e_n\left(X_t^i\right) \circ d\beta_n(t).$$

Without noise, there are (explicit) examples of initial conditions $\left(X_0^1, \dots, X_0^N\right)$ and intensities $\left(a_1, \dots, a_N\right)$ such that collision of vortices occurs in finite time (blow-up).

Theorem (F.-Gubinelli-Priola, SPA 2015)

There exists (σ_n) *such that for every* $\left(X_0^1, ..., X_0^N\right)$ *and* $(a_1, ..., a_N)$ *collision does not happen, with probability one.*

With Delarue and Vincenzi we have proved an analogous result for another nonlinear PDE, the so called Vlasov-Poisson equation, plus additional results.

At the end, it is meaningful to ask ourselves whether such space-dependent noises regularize also the L^p theory of Euler equation.