Stochastic Partial Differential Equations I

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Introduction

Except for isolated contributions (but relevant, like P.L. Chow, Vishik-Fursikov, Bensoussan-Temam), I think it was an idea of J.L. Lions and A. Bensoussan in the seventies to ask young very clever PhD students to develop a coherent theory of SPDEs.

They identified two initial branches, equations covered by monotonicity methods and those by the compactness methods.

E. Pardoux wrote his thesis on the monotonicity method, a masterpiece of incredible completeness also from the viewpoint of stochastic integration in infinite dimensions, just overcome later on by Krylov and Rozovski for a little detail of generalization. Recently the subject has been revised by Prevôt-Roeckner, B. Gess and others since the area of monotone operators is wider than the one spanned by Pardoux.

The compactness method was in the thesis of Viot, later considered again by Metivier and many others.

It is a very flexible method and perhaps for this reason it escaped such a complete solution as the one in Pardoux thesis.

I will review some elements of the compactness method below.

Slightly later on, say in the eighties, Da Prato and Zabczyk and their groups (and later on other groups) developed the semigroup approach.

It is also very flexible and in a sense the best one to extend finite dimensional results; it has been very successful also for stochastic control theory.

Other methods have been also developed, like Walsh one, and methods based on infinite dimensional calculus like Hida calculus and similar. More recently there are specific methods for other classes of PDEs, like fully nonlinear (P.L. Lions and Souganidis), dispersive equations and conservation laws (Debussche et al).

Nowadays it is difficult to identify a PDE which was not already considered with some kind of noise and for which a theorem of existence has not been proved. For non-extreme SPDEs, open existence questions are not common anymore.

But there are "extreme SPDEs", called often "singular SPDEs". They arise in Mathematical Physics, precisely in Kardar-Parisi-Zhang theory of random interfaces, in Quantum Field theory, and in other specific problems (Anderson model, for instance).

They have very singular noise which apprently is not compatible with the space-dimension and regularity theory of the PDE part. Hairer and coworkers, and just a little later Gubinelli and coworkers, revolutioned this sector.

Plan of the lectures

- The method of compactness
- Kolmogorov equations
- Open problems of regularization by noise

The first subject is introduced with the aim of teaching something classical but always extremely useful.
The second subject is still relatively new, rich of open questions and potential and related to advanced research activity in Bologna.
The third subject provides examples of quite advanced questions of current research.

Two open questions in fluid mechanics

Before we start with elements of the compactness method let us mention two open problems.

The first one, in a sense the most important one in the field, is the well posedness of the 3D Navier-Stokes equations:

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p &= \nu \Delta u \\ \operatorname{div} u &= 0 \end{aligned}$$

with suitable boundary and initial conditions. Existence of global weak solutions is known, but their uniqueness is open. Existence and uniqueness of local regular solutions from regular initial conditions is known, but their globality in time or conversely their blow-up is open.

A natural question is: can we find a noise such that the stochastic 3D Navier-Stokes equations are well posed (either in the class of weak solutions or the regular ones)?

The second problem is the well posedness of Euler equations; here there are questions at all levels (incompressible, compressible, d=2,3) so we choose the simplest one to formulate.
For incompressible 2D fluids one can introduce the variable "vorticity", $\omega = \nabla^{\perp} \cdot u$ and write the equations in the transport form

$$\partial_t \omega + u \cdot \nabla \omega = 0$$

(it is nonlinear, because u depends on ω). When $\omega(0)$ is bounded measurable, it is known that there is one and only one L^{∞}-solution.
But when $\omega(0) \in L^p$, with $p < \infty$, only existence of global L^p-solutions is known; uniqueness is open.
Can we find a noise such that the stochastic 2D Euler equations are well posed in the class of L^p-vorticity?

Method of compactness

Flexible and universal. Let us describe it in a particular case.

$$V \overset{\text{compact}}{\subset} H \subset Y$$

dense continuous injections. Aubin-Lions lemma:

$$L^2(0,T;V) \cap W^{1,1}(0,T;Y) \overset{\text{compact}}{\subset} L^2(0,T;H).$$

Several refinements, see J. Simon **AMPA** 1987, which also provide

$$\overset{\text{compact}}{\subset} C([0,T];H)$$

under stronger assumptions (see below).

If we have a sequence of functions (u_n) (usually solutions of an approximate equation) such that

$$\int_0^T \|u_n(t)\|_V^2 \, dt + \int_0^T \left\| \frac{du_n(t)}{dt} \right\|_Y dt \leq C$$

then there exists a subsequence (u_{n_k}) and a function $u \in L^2(0, T; H)$ such that

$$\lim_{n\to\infty} \int_0^T \|u_n(t) - u(t)\|_H^2 \, dt = 0.$$

Moreover, $u \in L^2(0, T; V)$ and (u_{n_k}) can be chosen so that it converges weakly to u in $L^2(0, T; V)$. Using these facts, if u_n was a solution of an approximating equation, often one can show that u is a solution of the limit equation.

Example

Assume $A : V \to V'$ linear bounded, such that there is $\nu > 0$ with

$$\langle Av, v \rangle_{V',V} \geq \nu \left\| v \right\|_V^2$$

for every $v \in V$; $B : V \times V \to V'$, bilinear continuous, such that

$$\langle B(v, v), v \rangle_{V',V} = 0$$

for every $v \in V$. Consider the equation:

$$\frac{du}{dt} + Au + B(u, u) = 0, \qquad u|_{t=0} = u_0$$

as an identity in V', or in the weak sense, for every $\phi \in V$,

$$\langle u(t), \phi \rangle_H + \int_0^t \langle Au(s), \phi \rangle_{V',V} \, ds$$
$$+ \int_0^t \langle B(u(s), u(s)), \phi \rangle_{V',V} \, ds = \langle u_0, \phi \rangle_H .$$

Assume H separable, $\{e_n\}$ a c.o.s. of H with $e_n \in V$, $\pi_n x = \sum_{i=1}^{n} \langle x, e_n \rangle_H e_n$, $H_n = \pi_n(H)$, $u_n \in C^1([0,T]; H_n)$ unique solution of

$$\frac{du_n}{dt} + \pi_n A u_n + \pi_n B(u_n, u_n) = 0, \qquad u_n|_{t=0} = \pi_n u_0$$

It is easy to prove

$$\begin{aligned} \sup_{t\in[0,T]} \|u_n(t)\|_H^2 + \int_0^T \|u_n(t)\|_V^2 \, dt &\leq C \\ \int_0^T \left\| \frac{du_n(t)}{dt} \right\|_{V'} dt &\leq C \end{aligned}$$

hence Aubin-Lions lemma applies ($Y = V'$) and we get (u_{n_k}) strongly convergent to some u in $L^2(0,T;H)$, and weakly in $L^2(0,T;V)$ (and weak-star in $L^\infty(0,T;H)$).

In the identity

$$\begin{aligned} & \langle u_n(t), \phi \rangle_H + \int_0^t \langle \pi_n A u_n +, \phi \rangle_{V',V} \, ds + \int_0^t \langle \pi_n B(u_n, u_n), \phi \rangle_{V',V} \, ds \\ = \; & \langle \pi_n u_0, \phi \rangle_H \end{aligned}$$

one can pass to the limit, with some additional argument, in all terms except for

$$\int_0^t \langle \pi_n B(u_n(s), u_n(s)), \phi \rangle_{V',V} \, ds.$$

It is here, in examples, that we take advantage of the strong convergence in $L^2(0, T; H)$. At the abstract level, let us assume that: if $v_n \to v$ strongly in H and weakly in V, then

$$\langle B(v_n, v_n), \phi \rangle_{V',V} \to \langle B(v, v), \phi \rangle_{V',V}$$

for all ϕ in a dense set of V. Then one can pass to the limit also in the quadratic term.

Navier-Stokes example

Let me describe the case of the 2D Navier-Stokes equation on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, for simplicity (all what I say can be extended to bounded domains with Dirichlet boundary conditions and to the 3D case, but with some care):

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p &= \nu \Delta u \\ \operatorname{div} u &= 0 \end{aligned}$$

(u=velocity, p=pressure), $\mathcal{D}$ = space of C^∞ vector fields $v : \mathbb{T}^2 \to \mathbb{R}^2$, periodic, mean zero, $\operatorname{div} v = 0$,

$$\begin{aligned} H &= \text{closure of } \mathcal{D} \text{ in the } L^2\text{-topology} \\ V &= \text{closure of } \mathcal{D} \text{ in the } W^{1,2}\text{-topology} \end{aligned}$$

$$\begin{aligned} Av &= \nu \Delta v \\ \langle B(u, v), z \rangle_{V',V} &= \int_{\mathbb{T}^2} (u(x) \cdot \nabla v(x)) \cdot z(x)\, dx \end{aligned}$$

All properties are satisfied; let us only discuss: if $v_n \to v$ strongly in H and weakly in V, then

$$\langle B(v_n, v_n), \phi \rangle_{V',V} \to \langle B(v, v), \phi \rangle_{V',V}$$

for all ϕ in a dense set of V.
We have

$$\begin{aligned} \langle B(v_n, v_n), \phi \rangle_{V',V} &= \int_{\mathbb{T}^2} (v_n(x) \cdot \nabla v_n(x)) \cdot \phi(x)\, dx \\ &= -\int_{\mathbb{T}^2} (v_n(x) \cdot \nabla \phi(x)) \cdot v_n(x)\, dx \end{aligned}$$

hence it is trivial, for $\phi \in C^1(\mathbb{T}^2, \mathbb{R}^2)$.

Theorem

For the 2D Navier-Stokes equation on $\mathbb{T}^2$, for every $u_0 \in H$ there exists one and only one solution of class

$$C\left([0, T]; H\right) \cap L^2\left(0, T; V\right).$$

Existence in $L^\infty\left(0, T; H\right) \cap L^2\left(0, T; V\right)$ has been shown; an additional argument (here omitted) is needed for $C\left([0, T]; H\right)$. Uniqueness, up to rigorous details, is due to the following estimates: if u_1, u_2 are two solutions, $v = u_1 - u_2$ satisfies

$$\frac{dv}{dt} + Av + B\left(u_1, v\right) + B\left(v, u_2\right) = 0, \qquad v|_{t=0} = 0.$$

From $\frac{dv}{dt} + Av + B(u_1, v) + B(v, u_2) = 0$ we get

$$\frac{1}{2}\frac{d}{dt}\|v\|_H^2 + \langle Av, v\rangle_{V',V} + \langle B(u_1, v), v\rangle_{V',V} + \langle B(v, u_2), v\rangle_{V',V} = 0$$

$$\frac{1}{2}\frac{d}{dt}\|v\|_H^2 + \nu\|v\|_V^2 \leq \left|\langle B(v, u_2), v\rangle_{V',V}\right|$$

$$\begin{aligned}\left|\int_{\mathbb{T}^2} (v(x)\cdot\nabla u_2(x))\cdot v(x)\,dx\right| &\leq C\|u_2\|_V \left(\int_{\mathbb{T}^2} |v(x)|^4\,dx\right)^{1/2} \\ &\leq C\|u_2\|_V \|v\|_H \|v\|_V\end{aligned}$$

$$\|v(t)\|_H^2 \leq \int_0^t \|u_2(s)\|_V^2 \|v(s)\|_H^2\,ds$$

hence $\|v(t)\|_H^2 = 0$ because $\int_0^t \|u_2(s)\|_V^2\,ds < \infty$.

Fractional Sobolev spaces

Essential for the stochastic case is the generalization (see J. Simon 1987, Corollary 5):

$$L^r(0,T;V)\cap W^{\alpha,p}(0,T;Y)\overset{\text{compact}}{\subset}L^r(0,T;H)$$

$$\text{if } \alpha p > 1-\frac{p}{r}\quad (p,r\geq 1)$$

Here $\alpha\in(0,1)$ and $W^{\alpha,p}(0,T;Y)$ is the space of functions $f\in L^p(0,T;Y)$ such that

$$\int_0^T\int_0^T\frac{\|f(t)-f(s)\|_Y^p}{|t-s|^{1+\alpha p}}dsdt<\infty.$$

Recall also that $W^{\alpha,p}(0,T;Y)\subset C([0,T];Y)$ if $\alpha p>1$.

Stochastic case

Let (μ_n) be a family of probability measures on Borel sets of $L^2(0,T;V) \cap W^{\alpha,p}(0,T;Y)$ with the following property: for every $\epsilon > 0$ there is a bounded set $B \subset L^2(0,T;V) \cap W^{\alpha,p}(0,T;Y)$ such that

$$\mu_n(B) \geq 1 - \epsilon$$

for every n. Then (μ_n) is tight in $L^2(0,T;H)$ because B is relatively compact in $L^2(0,T;H)$. Hence (Prohorov Thm) there exists a subsequence $\left(\mu_{n_k}\right)$ and a probability measure μ on Borel sets of $L^2(0,T;H)$ such that

$$\int_{L^2(0,T;H)} \Phi d\mu_{n_k} \to \int_{L^2(0,T;H)} \Phi d\mu$$

for every $\Phi : L^2(0,T;H) \to \mathbb{R}$ continuous bounded.

Typical case: μ_n is the law, on $L^2(0,T;V) \cap W^{\alpha,p}(0,T;Y)$ of a stochastic process $u_n(t)$:

$$\mu_n = \mathcal{L}(u_n).$$

This is why we have generalized to $\alpha \in (0,1)$: usual stochastic processes (like Brownian motion) do not have paths of class $W^{1,p}(0,T;Y)$ but only $W^{\alpha,p}(0,T;Y)$ for relatively small α ($\alpha \in \left(0,\frac{1}{2}\right)$ for Brownian motion).
Typical bounded sets in $L^2(0,T;V) \cap W^{\alpha,p}(0,T;Y)$: the set of all f such that

$$\int_0^T \|f(t)\|_V^2 \, dt + \int_0^T \int_0^T \frac{\|f(t) - f(s)\|_Y^p}{|t-s|^{1+\alpha p}} dsdt \leq C.$$

Typical way to prove $\mu_n(B) \geq 1 - \epsilon$:

$$\mathbb{E} \int_0^T \|u_n(t)\|_V^2 \, dt + \mathbb{E} \int_0^T \int_0^T \frac{\|u_n(t) - u_n(s)\|_Y^p}{|t-s|^{1+\alpha p}} dsdt \leq C$$

(by Chebyshev inequality).

Example

As above we have $A : V \to V'$, $B : V \times V \to V'$ and now the equation:

$$\frac{du}{dt} + Au + B(u, u) = \xi, \qquad u|_{t=0} = u_0$$

where ξ is white noise in time, with some covariance in space. Let us first define ξ.

Given a c.o.s. $\{e_n\}$ of H, given a probability space $(\Omega, \mathcal{F}, P)$, given a sequence of independent Brownian motions $\{\beta_n\}$ and nonnegative numbers $\{\sigma_n\}$, we introduce the formal series

$$W_t = \sum_{n=1}^{\infty} \sigma_n \beta_n(t)\, e_n.$$

If $\sum_{n=1}^{\infty} \sigma_n^2 < \infty$, it converges in $L^2(\Omega; H)$ for every $t \geq 0$:

$$\mathbb{E}\left[\|W_t\|_H^2\right] = \sum_{n=1}^{\infty} \sigma_n^2 \mathbb{E}\left[|\beta_n(t)|^2\right] = \sum_{n=1}^{\infty} \sigma_n^2 < \infty.$$

It defines an H-valued Brownian motion $(W_t)_{t \geq 0}$. We formally take $\xi = dW_t/dt$.

When $\sum_{n=1}^{\infty} \sigma_n^2 = +\infty$, but $\sigma_n^2 \leq C$, we may still define W_t as an element of $L^2(\Omega; Y)$ for some space $Y \supset H$. We omit the details. The meaning of equation

$$\frac{du}{dt} + Au + B(u, u) = \xi, \qquad u|_{t=0} = u_0$$

is (recall that formally $\xi = \sum_{n=1}^{\infty} \sigma_n \beta_n' e_n$)

$$\langle u(t), \phi \rangle_H - \langle u_0, \phi \rangle_H + \int_0^t \langle Au + B(u, u), \phi \rangle_{V', V} \, ds = \sum_{n=1}^{\infty} \sigma_n \beta_n(t) \langle e_n, \phi \rangle$$

and it is sufficient that, given ϕ in a dense set of V, $\sum_{n=1}^{\infty} \sigma_n \beta_n(t) \langle e_n, \phi \rangle_H$ converges in $L^2(\Omega; \mathbb{R})$, which is true if $\sigma_n^2 \leq C$:

$$\mathbb{E}\left[\left(\sum_{n=1}^{\infty} \sigma_n \beta_n(t) \langle e_n, \phi \rangle_H\right)^2\right] = \sum_{n=1}^{\infty} \sigma_n^2 \langle e_n, \phi \rangle_H^2 \leq C \|\phi\|_H^2 .$$

For simplicity let us study only the case $\sum_{n=1}^{\infty} \sigma_n^2 < \infty$. Similarly to the deterministic case one considers finite dimensional approximations. Assume H separable, $\{e_n\}$ a c.o.s. of H with $e_n \in V$, $\pi_n x = \sum_{i=1}^{n} \langle x, e_n \rangle_H e_n$, $H_n = \pi_n(H)$, $u_n \in C^1([0,T]; H_n)$ unique solution of

$$du_n + \pi_n \left(Au_n + B(u_n, u_n) \right) dt = \sum_{k=1}^{n} \sigma_k d\beta_k(t) e_n$$

with $u_n|_{t=0} = \pi_n u_0$, which has a unique continuous adapted solution. By Itô formula

$$\frac{1}{2} d \|u_n\|_H^2 + \nu \|u_n\|_V^2 = \sum_{k=1}^{n} \sigma_k \langle u_n, e_k \rangle_H d\beta_k(t) + \frac{1}{2} \sum_{k=1}^{n} \sigma_k^2 dt$$

which easily implies

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|u_n(t)\|_H^2 + \int_0^T \|u_n(t)\|_V^2 dt \right] \le C.$$

Recall above that we would like to have also (for $\alpha p > 1 - \frac{p}{2}$ and for a suitable space $Y \supset H$)

$$\int_0^T \int_0^T \frac{\mathbb{E}\left[\left\| u_n(t) - u_n(s) \right\|_Y^p\right]}{\left|t-s\right|^{1+\alpha p}} dsdt \leq C.$$

But $(t \geq s \geq 0)$

$$\begin{aligned} u_n(t) - u_n(s) &= -\int_s^t \pi_n \left(Au_n + B(u_n, u_n)\right) dr \\ &\quad + \sum_{k=1}^n \sigma_k \left(\beta_k(t) - \beta_k(s)\right) e_n. \end{aligned}$$

Choose for instance $p = 2$, $Y \supset V'$,

$$
\begin{aligned}
& \left\| u_n(t) - u_n(s) \right\|_Y^2 \\
\leq \quad & (t-s) \int_0^T \left\| \pi_n A u_n + \pi_n B(u_n, u_n) \right\|_Y^2 dr + C \sum_{k=1}^n \sigma_k^2 \left(\beta_k(t) - \beta_k(s) \right)^2
\end{aligned}
$$

$$
\begin{aligned}
& \mathbb{E}\left[\left\| u_n(t) - u_n(s) \right\|_Y^2 \right] \\
\leq \quad & C(t-s) \left(\int_0^T \mathbb{E}\left[\left\| u_n \right\|_V^2 + \left\| B(u_n, u_n) \right\|_Y^2 \right] dr + \sum_{k=1}^n \sigma_k^2 \right).
\end{aligned}
$$

Comparing the requirement (for $\alpha p > 1 - \frac{p}{2}$ and for a suitable space $Y \supset H$)

$$\int_0^T \int_0^T \frac{\mathbb{E}\left[\|u_n(t) - u_n(s)\|_Y^p\right]}{|t-s|^{1+\alpha p}} dsdt \leq C$$

with the choice $p = 2$, $Y \supset V'$ and the bound

$$\mathbb{E}\left[\|u_n(t) - u_n(s)\|_Y^2\right]$$

$$\leq C(t-s)\left(\int_0^T \mathbb{E}\left[\|u_n\|_{V'}^2 + \|B(u_n, u_n)\|_Y^2\right] dr + \sum_{k=1}^n \sigma_k^2\right)$$

we see we only need

$$\int_0^T \mathbb{E}\left[\|B(u_n, u_n)\|_Y^2\right] dr \leq C$$

and then we can choose any $\alpha \in \left(0, \frac{1}{2}\right)$.

The proof of this bound requires additional structure of B, satisfied for instance in the Navier-Stokes case, namely that there exists a space $Y \supset V'$ such that

$$\|B(v,v)\|_Y^2 \leq C \|v\|_H^4 .$$

This property is satisfied by the operator

$$\langle B(u,v), z\rangle_{V',V} = \int_{\mathbb{T}^2} (u(x) \cdot \nabla v(x)) \cdot z(x)\, dx$$

by taking $Y = C^1\left(\mathbb{T}^2, \mathbb{R}^2\right)'$:

$$\begin{aligned} \|B(v,v)\|_Y &= \sup_{\|\phi\|_{C^1} \leq 1} |\langle B(v,v), \phi\rangle| = \\ \sup_{\|\phi\|_{C^1} \leq 1} \left| \int_{\mathbb{T}^2} (v(x) \cdot \nabla \phi(x)) \cdot v(x)\, dx \right| &\leq C \|v\|_H^2 . \end{aligned}$$

The additional property

$$\|B(v,v)\|_Y^2 \leq C \|v\|_H^4$$

implies that, in order to fulfill

$$\int_0^T \mathbb{E}\left[\|B(u_n,u_n)\|_Y^2\right] dr \leq C$$

it is sufficient to prove

$$\int_0^T \mathbb{E}\left[\|u_n\|_H^4\right] dr \leq C.$$

Above, we only proved $\mathbb{E}\left[\sup_{t\in[0,T]} \|u_n(t)\|_H^2\right] \leq C$. In order to reach power 4 it is sufficient to repeat the computations using Doob's inequality.

Finally we have proved

$$\mathbb{E}\int_0^T \|u_n(t)\|_V^2\, dt + \mathbb{E}\int_0^T\int_0^T \frac{\|u_n(t)-u_n(s)\|_Y^2}{|t-s|^{1+\alpha 2}} dsdt \leq C$$

for a suitable space Y and for any $\alpha \in \left(0, \frac{1}{2}\right)$. By the previous compactness properties this implies that the family of laws

$$\mu_n = \mathcal{L}(u_n)$$

is tight in $L^2(0, T; H)$.

By Prohorov theorem, there exists a subsequence $\left(\mu_{n_n}\right)$ which converges weakly to some probability measure μ on $L^2(0, T; H)$. It remains to identify μ as the law of of a process u which solves the stochastic Navier-Stokes equation. This last part of the procedure is quite technical, based on Skorohod theorem and martingale theory and is thus omitted here for reasons of time; one can find it in many papers and books (like Da Prato - Zabczyk).

Theorem

For the stochastic 2D Navier-Stokes equation on $\mathbb{T}^2$, with additive noise and $\sum_{n=1}^{\infty}\sigma_n^2<\infty$, for every $u_0\in H$ there exists one and only one H-continuous adapted solution with the property

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|u(t)\|_H^2+\int_0^T\|u(t)\|_V^2\,dt\right]\le C.$$

The uniqueness statement is easy: if u_1,u_2 are two solutions, $v=u_1-u_2$ satisfies the deterministic equation

$$\frac{dv}{dt}+Av+B(u_1,v)+B(v,u_2)=0,\qquad v|_{t=0}=0$$

hence, exactly with the same computations of the deterministic case we get

$$\|v(t)\|_H^2\le\int_0^t\|u_2(s)\|_V^2\,\|v(s)\|_H^2\,ds$$

and thus $v(t)=0$, because $\int_0^t\|u_2(s)\|_V^2\,ds<\infty$ with probability one.

Having pathwise uniqueness, one can implement a method of Gyongy and Krylov, PTRF 1996, to prove also existence on the original probability space.

Indeed, what was not remarked above is that the method of compactness, due to the last step of Skorohod theorem, provides existence of a solution on a new probability space, a priori; the so called weak or martingale solutions, which are not necessarily adapted to the corresponding Brownian motion.

This difficuty disappears by the method of Gyongy and Krylov, when pathwise uniqueness is known. Alternatively, the classical argument is using Yamada-Watanabe theorem.

These difficulties come from the fact that compactness of stochastic processes is not a natural question from the viewpoint of the variable $\omega \in \Omega$. Compactness of the laws is the surrogate, but then one has to reconstruct a process.

Malliavin calculus may provide an alternative, proving compactness also in $\omega \in \Omega$. There are results in this direction but rarely used. Maybe this aspect could be improved.

Singular noise

When $\sum_{n=1}^{\infty} \sigma_n^2 = +\infty$ we are in the direction of the singular SPDEs mentioned in the introduction.
Here, precisely in space dimension 2, one has to distinguish between the case

$$\sigma_n^2 = 1$$

called cylindrical noise, or space-time white noise, and intermediate cases between this and $\sum_{n=1}^{\infty} \sigma_n^2 < \infty$.
Precisely, the operator $-A$ has eigenvalues (λ_n), $0 < \lambda_1 < \lambda_2 < \ldots$, $\lim_{n\to\infty} \lambda_n = +\infty$ and one can consider the case

$$\sigma_n^2 = \lambda_n^{-\alpha}.$$

It turns out $(d = 2)$ that for $\alpha \leq 1$ one has $\sum_{n=1}^{\infty} \sigma_n^2 = +\infty$, while for $\alpha > 1$ one has $\sum_{n=1}^{\infty} \sigma_n^2 < \infty$.
The intermediate case is

$$0 < \alpha \leq 1.$$

In this case one can use the following method: solve the linear equation (Stokes equation)

$$\frac{dz}{dt} + Az + \xi, \qquad z|_{t=0} = 0$$

and consider the equation for

$$v = u - z$$

which is

$$\frac{dv}{dt} + Av + B(v + z, v + z) = 0, \qquad v|_{t=0} = u_0. \tag{1}$$

This is a deterministic PDE depending on the random "function" $z(t)$. From the explicit formula

$$z(t) = \sum_{n=1}^{\infty} \sigma_n \int_0^t e^{-\lambda_n (t-s)} d\beta_n(s)\, e_n$$

and the assumption $\sigma_n^2 = \lambda_n^{-\alpha}$, $\alpha > 0$, one can prove that z is a true function, with sufficient regularity to study equation (1) by classical methods.
When $\sigma_n^2 = 1$, $z(t)$ takes values in a space of distributions $W^{-\epsilon,2}$. The meaning of the quadratic term when applied to distributions has to be clarified. Using Gaussian renormalization (Wick products) it has a meaning and the final equation can be solved.
This was a breakthrough of Da Prato and Debussche, followed by additional fundamental results of Albeverio and Ferrario. Some people say it was the beginning of the theory of singular SPDEs, the germ of the idea of regularity structures.

Embedding into continuous functions

The property of continuity in time of paths sometimes follows a posteriori, from the SPDE.
Alternatively, we may try to prove convergence of the approximating scheme in the uniform topology. To this purpose we may use the following result of J. Simon 1987, Corollary 9: assume in addition ($\theta \in (0,1)$)

$$\begin{aligned} \|v\|_H &\leq C \|v\|_V^{1-\theta} \|v\|_Y^{\theta} \qquad \theta \in (0,1) \\ \alpha p &> 1 \text{ and } r > \frac{1-\theta}{\theta} \frac{p}{\alpha p - 1} \quad (p, r \geq 1). \end{aligned}$$

Then

$$L^r(0,T;V) \cap W^{\alpha,p}(0,T;Y) \overset{\text{compact}}{\subset} C([0,T];H).$$

In stochastic examples one has to prove

$$\mathbb{E}\int_0^T\int_0^T \frac{\left\| u_n(t)-u_n(s)\right\|_Y^p}{|t-s|^{1+\alpha p}}dsdt \leq C$$

for some $\alpha p > 1$ and then

$$\mathbb{E}\int_0^T \left\| u_n(t)\right\|_V^r dt \leq C$$

for r arbitrarily large.

The power of this technique should be compared with criteria based on Ascoli-Arzelà. They require for instance

$$\mathbb{E}\left[\sup_{0\leq s<t\leq T}\frac{\left\|u_n\left(t\right)-u_n\left(s\right)\right\|_H}{\left|t-s\right|^{\epsilon}}\right]\leq C$$

but it is very difficult to estimate the supremum in two time parameters. Aldous criterium based on stopping times and martingale theory is a great advance in this direction. The previous method based on J. Simon theorem is an alternative.

In Probability the case of càdlàg processes is very important. Skhorokod space and topology and corresponding compactness criteria are well developed for them. An open problem is whether one can build criteria similar to those above, suitable for càdlàg processes.

The method of compactness works well also to prove convergence of particle systems to (S)PDEs.

In the purely mean field cases, it is not competitive with other approaches like those described in the book of Sznitman, more quantitative, easier etc.

In variants of the mean field case the compactness method shows its flexibility.

The 3D case

With relatively minor modifications with respect to the 2D case, one can also prove:

Theorem

For the stochastic 3D Navier-Stokes equation on $\mathbb{T}^3$, with additive noise and $\sum_{n=1}^{\infty} \sigma_n^2 < \infty$, for every $u_0 \in H$ there exists at least one H-weakly continuous adapted solution with the property

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|u(t)\|_H^2 + \int_0^T \|u(t)\|_V^2 \, dt\right] \leq C.$$

The solution is understood in the so called weak probabilistic sense: there exists a probability space $(\Omega, \mathcal{F}, P)$, a filtration $(\mathcal{F}_t)$, an $(\mathcal{F}_t)$-Brownian motion and a solution adapted to $(\mathcal{F}_t)$ (but not necessarily to the Brownian motion). Otherwise, there exist a path-by-path solution, on the original space, but we do not know whether it is adapted. Uniqueness remains open, in spite of formidable efforts (more later).

In the case

$$\partial_t \omega + u \cdot \nabla \omega = 0$$

we do not have the operator A, which provided the bounds in V. However, considering $u(t) \in H$ and $\omega(t) \in V$, the a priori estimate in V comes from the conservation of enstrophy:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2} \omega^2 dx &= \int_{\mathbb{T}^2} \omega \left(u \cdot \nabla \omega\right) dx = \frac{1}{2}\int_{\mathbb{T}^2} u \cdot \nabla \left(\omega^2\right) dx \\ &= -\frac{1}{2}\int_{\mathbb{T}^2} \operatorname{div} u \cdot \omega^2 dx = 0 \end{aligned}$$

so the method of compactness applies again!

It gives us:

Theorem

For the deterministic incompressible 2D Euler equations on $\mathbb{T}^2$*, for every* $\omega(0) \in L^2$ *there exists at least one* L^2*-weakly continuous solution, with the property* $\sup_{t\in[0,T]} \|\omega(t)\|_{L^2}^2 < \infty$.

Concerning stochastic perturbations, I do not find so natural (see below) to use an additive noise. More natural is the transport type noise

$$\partial_t \omega + u \cdot \nabla\omega + \sigma\nabla\omega \circ W' = 0.$$

The multiplication denoted by $\nabla\omega \circ W'$ is the Stratonovich one; we do not start discussing it here but only remark that the rules of calculus, for it, are the same of the rules of deterministic calculus (the price are more difficult proofs of such rules and more restrictive conditions for their validity).

With this kind of noise we again have conservation of enstrophy:

$$\frac{1}{2} d \int_{\mathbb{T}^2} \omega^2 dx = \sigma \int_{\mathbb{T}^2} \omega \nabla \omega dx \circ dW = \frac{\sigma}{2} \int_{\mathbb{T}^2} \nabla \left(\omega^2 \right) dx \circ dW = 0.$$

With the same compactness arguments explained above one can prove:

Theorem

For the stochastic incompressible 2D Euler equations on $\mathbb{T}^2$, for every $\omega(0) \in L^2$ there exists at least one L^2-weakly continuous adapted solution, with the property

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|\omega(t)\|_{L^2}^2 \right] \leq C.$$

Uniqueness remains open, even with more elaborate noises (more later).