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Lezioni di Analisi Matematica 2 e Complementi

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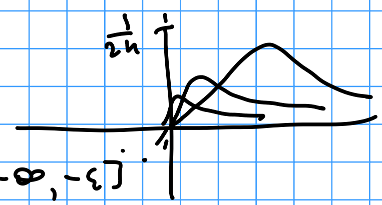
Ricevimento su appuntamento da concordare per email

$$f(x)=\sum_{n=1}^{\infty}\frac{x}{1+n^2x^2}$$

VISTO CHE – la serie conv. pt. su tutto $\mathbb{R}$

- Converge totalmente su $[\varepsilon,+\infty[$ $\forall\varepsilon>0$ (e $]-\infty,-\varepsilon]$ $\forall\varepsilon>0$)

  (NON CONV. TOT. SU $\mathbb{R}$)

  $\Rightarrow$ f è continua su $\mathbb{R}\setminus\{0\}$

  (è continua su $[\varepsilon,+\infty[\,\cup\,]-\infty,-\varepsilon]$ $\forall\varepsilon>0$)

- $\cancel{\lim_{x\to 0} f(x)=0}$ è falso $\Rightarrow$ Non c'è convergenza unif. su $\mathbb{R}$

- $\lim_{x\to+\infty} f(x) = 0$ (per la conv. unif. su $[1,+\infty[$)
  ($x\to+\infty$)

Vediamo se f è derivabile su $\mathbb{R}\setminus\{0\}$. Per questo uso il risultato sulle derivate. Devo calcolare f_n' e vedere se la serie $\sum f_n'$ conv. unif. o qualcosa.

Si ha: $$f_n'(x)=\frac{1+n^2x^2-x(n^2 2x)}{(1+n^2x^2)^2}=\frac{1-n^2x^2}{(1+n^2x^2)^2}$$

Cerco di dim. la conv. totale di $\sum f'_n$ su $[\varepsilon, +\infty[$ avendo fissato $\varepsilon > 0$. Mi serve dunque

$$\| f_n' \|_{\infty, [\varepsilon, +\infty[} = \sup_{x \geq \varepsilon} \frac{1 - n^2 x^2}{(1+n^2x^2)^2}$$ ← mi basta una stima

$$\leq \sup_{x \geq \varepsilon} \frac{1+n^2x^2}{(1+n^2x^2)^2} = \sup_{x\geq\varepsilon} \frac{1}{1+n^2x^2} \leq \frac{1}{1+n^2\varepsilon^2}$$

Dunque $\| f'_n \|_{\infty, [\varepsilon, +\infty[} \leq \frac{1}{1+\varepsilon^2 n^2}$ e so che $\sum \frac{1}{1+\varepsilon^2 n^2} < +\infty$

$\Rightarrow \sum \| f_n' \|_\infty < +\infty \Rightarrow$ Ho LA CONV. TOTALE di $\sum f_n'$

$\Rightarrow \sum f_n'$ conv. unif. su $[\varepsilon, +\infty[$

(e ricordo che $\sum f_n$ conv. unif. su $[\varepsilon, +\infty[$)

DUNQUE f è derivabile e

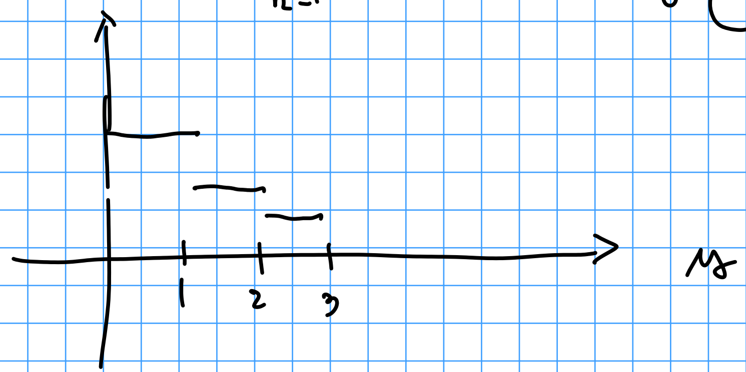
$$f' = \frac{d}{dx} \sum \frac{x}{1+n^2x^2} = \sum_{n=1}^{\infty} f_n' = \sum_{n=1}^{\infty} \frac{1-n^2x^2}{(1+n^2x^2)^2}$$

- Posso calcolare esplicitamente il limite di $f(x)$ per $x \to 0^+$

Infatti posso scrivere

$$f(x) = \sum_{n=1}^{\infty} \frac{x}{1+n^2x^2} = \int_0^{+\infty} \underbrace{\frac{x}{1+[y]^2x^2}}_{g(x,y)} dy$$

x fissato



$g(x,y) = f_n(x)$ se $n-1 \leq y \leq n$

$g(x,y) = f_1(x)$ se $0 \leq y < 1$

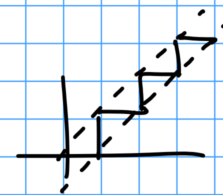
$= f_2(x)$ se $1 \leq y < 2$

$= f_3(x)$ se $2 \leq y < 3$ . . .

$\int g(x,y)\,dy = f_1(x) + f_2(x) + \dots$

$$\underbrace{\int_0^{+\infty} \frac{x \, dy}{1+y^2x^2}}_{} \leq \int_0^{+\infty} \frac{x}{1+[y]^2x^2}\,dy \leq \int_0^{+\infty} \frac{x}{1+(y-1)^2x^2}\,dy$$

$$y-1 \leq [y] \leq y$$



sostituzione

$z = yx \quad dz = x\,dy$

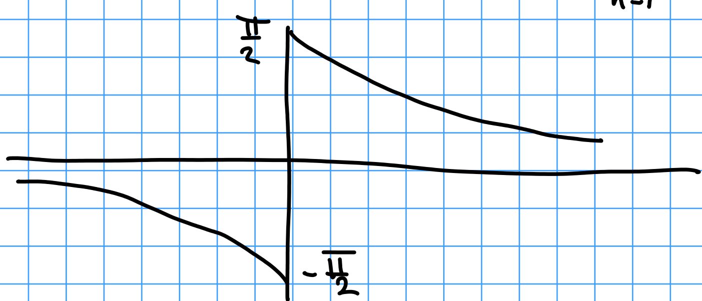
$$\int_0^{+\infty} \frac{dz}{1+z^2} = \arctan(z)\Big|_0^{+\infty} = \frac{\pi}{2}$$

Nell'integrale di destra sostituisco $z = (y-1)x \quad dz = x\,dy$

Trovo $\displaystyle\int_{-x}^{+\infty} \frac{dz}{1+z^2}$. QUINDI

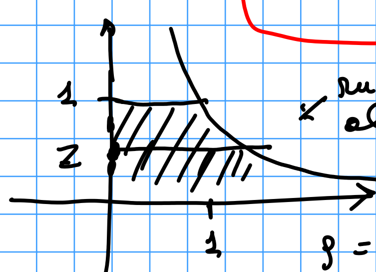
$$\int_0^{+\infty} \frac{dz}{1+z^2} \leq \sum_{n=1}^{\infty} \frac{x}{1+n^2x^2} \leq \underbrace{\int_{-x}^{+\infty} \frac{dz}{1+z^2}}_{\frac{\pi}{2}+\arctan(x)} \quad \forall x>0$$

Se mando $x \to 0^+$ trovo $\displaystyle\sum_{n=1}^{\infty} \frac{x}{1+n^2x^2} = \int_0^{+\infty} \frac{dz}{1+z^2} = \frac{\pi}{2}$



CALCOLARE L'INTEGRALE (al variare di $\alpha > 0$)

$$\iiint_D \frac{z}{(x^2+y^2)^{\alpha}} \quad \text{dove} \quad \boxed{D = \left\{ 0 \leq z \leq 1 \; , \; x^2+y^2 \leq \frac{1}{z^2} \right\}}$$

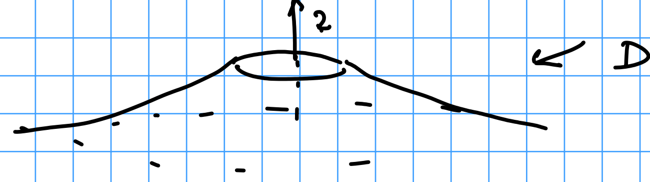


$\rho = \sqrt{x^2+y^2}$

$\rho^2 \leq \frac{1}{z^2}$

$\rho^2 z^2 \leq 1$

$0 \leq \rho z \leq 1$



Passiamo in "coordinate cilindriche" (ρ, θ, z')

$z = z'$ $\quad x = \rho\cos\theta \quad y = \rho\sin\theta$

$\det J = \rho \, d\rho \, d\theta \, dz'$

TROVO L'INTEGRALE $\displaystyle\iiint_{D'} \frac{z}{(\rho z)^{\alpha}} \rho \, d\rho \, d\theta \, dz :=$

$D' = \{ 0 \le z \le 1, \ \rho \le 1/z, \ 0 \le \theta \le 2\pi \}$

$$2\pi \int_0^1 z \left(\int_0^{\frac{1}{z}} \rho^{1-2\alpha} \, d\rho \right) dz \quad (*)$$

(può essere $+\infty$
STIAMO INTEGRANDO UNA FUNZIONE MISURABILE E POSITIVA)

oppure $$2\pi \left(\int_0^1 \left(\int_0^1 \rho^{1-2\alpha} z \, dz \right) d\rho + \int_1^{+\infty} \left(\int_0^{1/\rho} \rho^{1-2\alpha} z \, dz \right) d\rho \right)$$

$$(*) = 2\pi \int_0^1 z \left[\frac{\rho^{2-2\alpha}}{2-2\alpha} \right]_0^{1/z} dz$$ ← È FINITO se $2 - 2\alpha > 0 \iff \boxed{\alpha < 1}$

se $2-2\alpha \neq 0$, se no trovo un logaritmo

$$\frac{\pi}{1-\alpha} \int_0^1 z \left(\frac{1}{z}\right)^{2-2\alpha} dz = \frac{\pi}{1-\alpha} \int_0^1 z^{1+2\alpha-2} \, dz = \frac{\pi}{1-\alpha} \int_0^1 z^{2\alpha-1} \, dz$$

$$= \frac{\pi}{1-\alpha} \left[\frac{z^{2\alpha}}{2\alpha} \right]_0^1$$ (se $2\alpha > 0 \quad \alpha > 0$) $= \dfrac{\pi}{2\alpha(1-\alpha)}$ $\quad \boxed{0 < \alpha < 1}$

Possiamo provare il secondo metodo

$$2\pi\left(\int_0^1\left(\int_0^1 \rho^{1-2\alpha} z\,dz\right)d\rho + \int_1^{+\infty}\left(\int_0^{1/\rho}\rho^{1-2\alpha} z\,dz\right)d\rho\right) =$$

$$2\pi\left(\int_0^1 \rho^{1-2\alpha}\left[\frac{z^2}{2}\right]_0^1 d\rho + \int_1^{+\infty}\rho^{1-2\alpha}\left[\frac{z^2}{2}\right]_0^{1/\rho} d\rho\right) =$$

$$\pi\left(\int_0^1 \rho^{1-2\alpha}\,d\rho + \int_1^{+\infty}\rho^{1-2\alpha}\frac{1}{\rho^2}\,d\rho\right)=$$

$$\pi\left(\int_0^1\rho^{1-2\alpha}\,d\rho+\int_1^{+\infty}\rho^{-1-2\alpha}\,d\rho\right)=$$

$$\pi\left(\left[\frac{\rho^{2-2\alpha}}{2-2\alpha}\right]_0^1 + \left[\frac{\rho^{-2\alpha}}{-2\alpha}\right]_1^{+\infty}\right)=$$

$2-2\alpha>0$ $\qquad$ $\alpha>0$ $\qquad$ $\boxed{0<\alpha<1}$

$$\pi\left(\frac{1}{2-2\alpha}+\frac{1}{2\alpha}\right)=\pi\,\frac{2\alpha+2-2\alpha}{2\alpha(2-2\alpha)}=\frac{\pi}{2\alpha(1-\alpha)}$$ torna con il calcolo precedente

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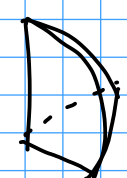
Calcolare $\displaystyle\iiint_D xyz\,dx\,dy\,dz$ $\qquad D=\{x^2+y^2+z^2\le 1,\ x\ge0,\ y\ge0,\ z\ge0\}$



D

Conviene usare le "coordinate sferiche"

$$\boxed{x=\rho\cos\theta\sin\varphi \qquad y=\rho\sin\theta\sin\varphi \qquad z=\rho\cos\varphi}$$

Se si fanno i conti ho

$|\det J\phi|(\rho,\theta,\varphi)=\rho^2\sin\varphi$ $\qquad$ Dunque l'integrale diventa

$$\iiint_{\substack{\rho\le1\\0\le\theta\le\pi/2\\0\le\varphi\le\pi/2}} (\rho\cos\theta\sin\varphi)(\rho\sin\theta\sin\varphi)(\rho\cos\varphi)\,\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi=$$

$$\underbrace{\int\int\int}\ \rho^5 \cos\theta \sin\theta \ \sin^3\varphi \cos\varphi \ d\rho\, d\theta\, d\varphi =$$

$\sin\varphi = s \quad \cos\varphi \, d\varphi = ds$

$$\int_0^{\pi/2} \cos\theta \sin\theta \, d\theta \int_0^{\pi/2} \sin^3\varphi \cos\varphi \, d\varphi \int_0^1 \rho^5 \, d\rho =$$

$$\frac{1}{4}\Big[-\cos(2\theta)\Big]_0^{\frac{\pi}{2}} \int_0^1 s^3\, ds \left[\frac{\rho^6}{6}\right]_0^1 =$$

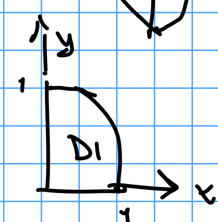
$$\frac{1}{4}\left(1+1\right)\left[\frac{s^4}{4}\right]_0^1 \frac{1}{6} = \frac{1}{2}\,\frac{1}{4}\,\frac{1}{6} = \frac{1}{48}$$

SI PUÒ FARE SENZA CAMBI DI VARIABILE

VEDENDO D come insieme NORMALE (RISPETTO ALL'ASSE Z)

$$\iiint_D xyz \, dx\, dy\, dz = \iint_{D_1} xy \int_0^{\sqrt{1-x^2-y^2}} z \, dz$$

dove $D_1 = \{ x^2+y^2 \leq 1, \quad x \geq 0, y \geq 0 \}$



$$= \iint_{D_1} xy \left[\frac{z^2}{2}\right]_0^{\sqrt{1-x^2-y^2}} dx\,dy = \frac{1}{2}\iint_{D_1} xy\,(1-x^2-y^2)$$

$$= \frac{1}{2}\int_0^1 x \left(\int_0^{\sqrt{1-x^2}} y(1-x^2-y^2) \right) dx =$$

$$\frac{1}{2}\int_0^1 x \left(\left[(1-x^2)\frac{y^2}{2}\right]_0^{\sqrt{1-x^2}} - \left[\frac{y^4}{4}\right]_0^{\sqrt{1-x^2}} \right) dx =$$

$$\frac{1}{2}\int_0^1 x \left(\frac{(1-x^2)(1-x^2)}{2} - \frac{(1-x^2)^2}{4} \right) dx$$

$$\frac{1}{8}\int_0^1 x\,(1-x^2)^2 \, dx = \qquad t = x^2 \qquad dt = 2x\,dx$$

$$\frac{1}{16}\int_0^1 (1-t)^2 \, dt = \frac{1}{16}\left[\frac{(1-t)^3}{3}(-1)\right]_0^1 = \frac{1}{16}\cdot\frac{1}{3} = \frac{1}{48}$$

Torna

$$\iint_D \frac{x^2-y^2}{1+x^2y^2}\,dx\,dy \qquad D=\{0\le x-y\le xy\le 1,\ x\geq 0\}$$

$\Downarrow$ $y\geq 0$

Suggerimenti: fare un cambio di variabile $\xi = x-y \quad \eta = xy$

Se seguo il suggerimento fo il cambio di variabile

$$(\xi,\eta)=\Phi(x,y)=\begin{pmatrix} x-y \\ xy \end{pmatrix} \Rightarrow$$

$$J_\Phi(x,y)=\begin{bmatrix} 1 & -1 \\ y & x \end{bmatrix} \Rightarrow \det J_\Phi(x,y) = x+y \geq 0$$

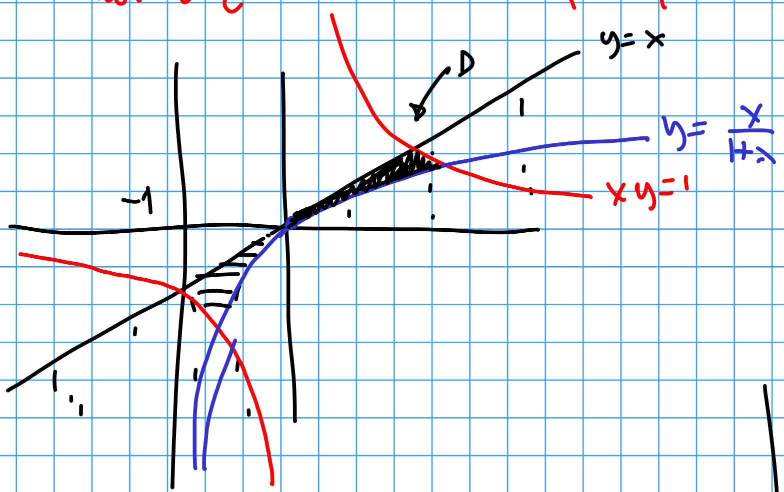
$$|J_\Phi(x,y)| = x+y$$

L'integrale diventa

$$\iint_D \frac{x^2-y^2}{1+x^2y^2}\,dx\,dy = \iint_D \frac{(x-y)}{1+x^2y^2}\overbrace{(x+y)\,dx\,dy}^{d\eta\, d\xi} =$$

$$\boxed{\iint_{D_1} \frac{\xi}{1+\eta^2}\,d\xi\,d\eta} \leftarrow \circledast \qquad D_1=\{0\le \xi\le \eta\le 1\}$$

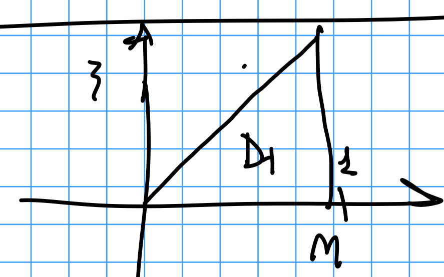
(se non c'è $x\geq 0$ in D, Φ non è iniettiva)



$$xy \geq x-y$$
$$y(1+x)\geq x$$
$$y \geq \frac{x}{1+x} \qquad (\text{se } 1+x\geq 0)$$

$$\circledast = \int_0^1 \left(\int_0^\eta \frac{\xi}{1+\eta^2}\,d\xi \right) d\eta =$$

D_1

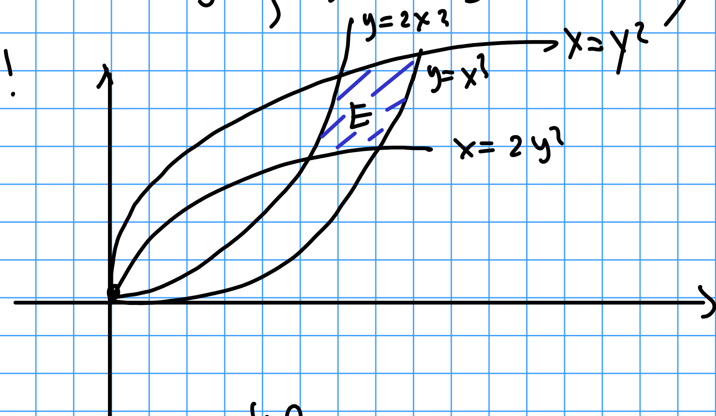


$$\int_0^1 \frac{1}{1+\eta^2}\left[\frac{\xi^2}{2}\right]_0^{\eta} d\eta = \frac{1}{2}\int_0^1 \frac{\eta^2}{1+\eta^2}\, d\eta = \frac{1}{2}\int_0^1 \left(1-\frac{1}{1+\eta^2}\right) d\eta =$$

$$\frac{1}{2}\left(1 - \Big[\arctan(\eta)\Big]_0^1\right) = \frac{1}{2}\left(1-\frac{\pi}{4}\right)$$

$$E = \{(x,y) : y^2 \leq x \leq 2y^2 \,,\; x^2 \leq y \leq 2x^2\}$$

Voglio l'area di E!



Introduco un cambio di variabile

$$\xi = \frac{x}{y^2} \qquad \eta = \frac{y}{x^2} \qquad \Big(D \text{ si trasforma in } D_1 = \{1 \leq \xi \leq 2,\; 1 \leq \eta \leq 2\}\Big)$$

$$(\eta, \xi) = \phi(x,y) = \left(\frac{x}{y^2}, \frac{y}{x^2}\right) \qquad (x,y) = \phi^{-1}(\xi,\eta)$$

$$|E| = \iint_D dx\,dy = \iint_{D_1} 1 \,\left|\det J_{\phi^{-1}}(\xi,\eta)\right| d\xi\, d\eta$$

$$D \underset{\phi^{-1}}{\overset{\phi}{\rightleftarrows}} D_1$$

Devo trovare $\phi^{-1}(\xi,\eta)$, cioè devo risolvere il sistema

$$\begin{cases} \dfrac{x}{y^2} = \xi \\[2mm] \dfrac{y}{x^2} = \eta \end{cases} \iff \begin{cases} x = y^2\xi \\ y = x^2\eta \end{cases} \iff \begin{cases} x = y^2\xi \\ y = (y^2\xi)^2\eta = y^4\xi^2\eta \end{cases}$$

$y = 0 \,/\quad 1 = y^3 \xi^2 \eta \qquad$ dunque $\qquad y = \xi^{-\frac{2}{3}} \eta^{-\frac{1}{3}}$

$$x = \xi^{-\frac{4}{3}} \eta^{-\frac{2}{3}} \xi = \xi^{-\frac{1}{3}} \eta^{-\frac{2}{3}}$$

$$\phi^{-1}(\xi, \eta) = \left(\xi^{-\frac{1}{3}} \eta^{-\frac{2}{3}} \,,\; \xi^{-\frac{2}{3}} \eta^{-\frac{1}{3}} \right)$$

$$J_{\phi^{-1}} = \begin{bmatrix} -\frac{1}{3} \xi^{-\frac{4}{3}} \eta^{-\frac{2}{3}} & -\frac{2}{3} \xi^{-\frac{1}{3}} \eta^{-\frac{5}{3}} \\ -\frac{2}{3} \xi^{-\frac{5}{3}} \eta^{-\frac{1}{3}} & -\frac{1}{3} \xi^{-\frac{2}{3}} \eta^{-\frac{4}{3}} \end{bmatrix}$$

$$\det J_{\phi^{-1}} = \frac{1}{9} \xi^{-2} \eta^{-2} - \frac{4}{9} \xi^{-2} \eta^{-2} = -\frac{1}{3} \xi^{-2} \eta^{-2}$$

DUNQUE

$$|E| = \int_1^2 \left(\int_1^2 \frac{1}{3} \frac{1}{\xi^2 \eta^2} \, d\eta \right) d\xi = \frac{1}{3} \left[-\xi^{-1} \right]_1^2 \left[-\eta^{-1} \right]_1^2 =$$

$$\frac{1}{3} \left(1 - \frac{1}{2} \right) \left(1 - \frac{1}{2} \right) = \frac{1}{12}$$