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Lezione 72 30/05/2023

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Ricevimento su appuntamento da concordare per email

$$A = \begin{bmatrix} 3 & -2 & 3 \\ 5 & -4 & 5 \\ 1 & -1 & 0 \end{bmatrix}$$

$x(t) = e^t \quad y(t) = e^t \quad z(t) = 0$

$$\begin{cases} e^t = 3e^t - 2e^t + 0 \\ e^t = 5e^t - 4e^t + 0 \\ 0 = e^t - e^t \end{cases} \qquad \text{TORNA}$$

(sol. trovata dopo)

$$\begin{cases} x' = 3x - 2y + 3z \\ y' = 5x - 4y + 5z \\ z' = x - y \end{cases} \qquad (E.O)$$

$$\begin{cases} x' = 3x - 2y + 3z + 7e^{-t} \\ y' = 5x - 4y + 5z + 15e^{-t} \\ z' = x - y + e^{-t} \end{cases} \qquad B(t) = \begin{pmatrix} 7 \\ 15 \\ 1 \end{pmatrix} e^{-t}$$

Trovare e^{tA}.

Polinomio caratteristico

$$p(\lambda) = \det \begin{bmatrix} 3-\lambda & -2 & 3 \\ 5 & -4-\lambda & 5 \\ 1 & -1 & -\lambda \end{bmatrix} = (3-\lambda)(4+\lambda)\lambda - 10 - 15 - \{(3-\lambda)(-5) + (-4-\lambda)3 + (-\lambda)(-10)\}$$

$$= (12 - \lambda - \lambda^2)\lambda - 25 - \{-15 + 5\lambda - 12 - 3\lambda + 10\lambda\} =$$

$$-\lambda^3 - \lambda^2 + 12\lambda - 25 - 12\lambda + 27 = -\lambda^3 - \lambda^2 + 2$$

VEDO CHE $P(1) = 0$. $\quad P(\lambda) = -(\lambda-1)(\lambda^2+2\lambda+2)$

$$\begin{array}{c|ccc|c} & -1 & -1 & 0 & 2 \\ 1 & & -1 & -2 & -2 \\ \hline & -1 & -2 & -2 & 0 \end{array}$$

$$\lambda^2+2\lambda+2 = (\lambda+1)^2+1$$

$$\lambda_{1,2} = -1 \pm i$$

TRE AUTOVALORI DISTINTI $\quad \lambda_1 = 1 \quad \lambda_{2,3} = -1 \pm i$

(quelli complessi compaiono a coppie coniugate)

TROVO GLI AUTOVETTORI

$\underline{\lambda=1}$

$$\begin{bmatrix} 3 & -2 & 3 \\ 5 & -4 & 5 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

salto un passaggio

$$\begin{cases} 2x - 2y + 3z = 0 \\ 5x - 5y + 5z = 0 \\ x - y - z = 0 \end{cases}$$

$$\begin{cases} x = y + z \\ 2y + 2z - 2y + 3z = 0 \\ 5y + 5z - 5y + 5z = 0 \end{cases} \qquad \begin{cases} x = y + z \\ z = 0 \\ 0 = 0 \end{cases} \qquad \boxed{x = y \quad z = 0}$$

Per esempio $\quad e_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\lambda = -1 + i$

$$\begin{bmatrix} 4-i & -2 & 3 \\ 5 & -3-i & 5 \\ 1 & -1 & 1-i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} (4-i)x - 2y + 3z = 0 \\ 5x - (3+i)y + 5z = 0 \\ x - y \quad (1-i)z = 0 \end{cases} \leftarrow$$

$$\begin{cases} 0 \quad (4-i-2)y + [-(4-i)(1-i)+3]z = 0 & \leftarrow -(4-i)\,\text{III}^\circ \text{ riga} \\ 0 \quad (2+i)y + 5iz = 0 & \leftarrow -5\,\text{III}^\circ \text{ riga} \\ x - y + (1-i)z = 0 \end{cases}$$

$$\begin{cases} x - y + (1-i)z = 0 \\ (2-i)y + 5iz = 0 \\ (2-i)y + 5iz = 0 \end{cases}$$

STESSA EQ.

$3 - (4-i)(1-i) =$

$3 - (4 - 1 - 5i) =$

3

$$\begin{cases} y-(1-i)z = x \\ (2-i)y+5iz = 0 \end{cases} \qquad z = -\frac{(2-i)}{5i}y$$

$$y+\frac{(1-i)(2-i)}{5i}y = x \rightarrow y+\frac{(2-1-3i)(-i)}{5} = x$$

$$\frac{5 \quad -i-3}{5}y = x \qquad \frac{2-i}{5}y = x \qquad y = \frac{5x}{2-i} =$$

$$z = -\frac{(2-i)(2+i)x}{5i} = \qquad y = \frac{5(2+i)\ x}{4+1} = (2+i)x$$

$$-\frac{5x}{5i} = ix$$

$$\boxed{\begin{matrix} y = (2+i)x \\ z = ix \end{matrix}} \qquad e_2 = \begin{pmatrix} 1 \\ 2+i \\ i \end{pmatrix}$$

CONTROLLO

$$\begin{bmatrix} 3 & -2 & 3 \\ 5 & -4 & 5 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2+i \\ i \end{bmatrix} = \begin{bmatrix} 3-2(2+i)+3i \\ 5-4(2+i)+5i \\ 1-(2+i) \end{bmatrix} = \begin{bmatrix} 3-4-2i+3i \\ 5-8-4i+5i \\ 1-2-i \end{bmatrix} = \begin{bmatrix} -1+i \\ -3+i \\ -1-i \end{bmatrix}$$

$$\stackrel{??}{=} (-1+i)\begin{bmatrix} 1 \\ 2+i \\ i \end{bmatrix} = \begin{bmatrix} -1+i \\ (-1+i)(2+i) \\ (-1+i)i \end{bmatrix} = \begin{bmatrix} -1+i \\ -2-1+i \\ -i-1 \end{bmatrix} = \begin{bmatrix} -1+i \\ -3+i \\ -i-1 \end{bmatrix}$$

✓ TORNA

Per motivi standard $\qquad e_3 = \overline{e}_2 = \begin{pmatrix} 1 \\ 2-i \\ -i \end{pmatrix}$

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2+i & 2-i \\ 0 & i & -i \end{bmatrix} \qquad J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1+i & 0 \\ 0 & 0 & -1-i \end{bmatrix}$$

Proviamo a calcolare M^{-1}

$$\det M = (2+i)(-i) - i\,(2-i) - \left[(2-i) - (2+i)\right] =$$
$$-i\,(2+i+2-i) - [-2i] =$$
$$-4i + 2i = -2i$$

$$M^{-1} = \frac{1}{-2i}\,\mathrm{cof}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2+i, & i \\ 1 & 2-i & -i \end{bmatrix} = \frac{i}{2}\begin{bmatrix} -4i & 2i & -2i \\ i & -i & -1+i \\ i & -i & 1+i \end{bmatrix} =$$

$$\left(\frac{i}{2}\begin{bmatrix} (2+i)(-i)-i(2-i), & -(-i-i), & 2-i-(2+i) \\ i & -i & -[(2-i)-1] \\ i & -i & 2+i-1 \end{bmatrix}\right)$$

$$\frac{i}{2}\begin{bmatrix} -4i & 2i & -2i \\ i & -i & -1+i \\ i & -i & 1+i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 & -2 & 2 \\ -1 & 1 & -i-1 \\ -1 & 1 & i-1 \end{bmatrix} = M^{-1}$$

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2+i & 2-i \\ 0 & i & -i \end{bmatrix} \qquad J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1+i & 0 \\ 0 & 0 & -1-i \end{bmatrix}$$

$$\left(M\,M^{-1} = \frac{1}{2}\begin{bmatrix} 2, & 0 & \\ & & \\ & & \end{bmatrix}\right)$$

Cerchiamo la sol. dell'eq. omogenea con

dato iniziale $Y_0 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$$Y(t) = e^{tA} Y_0 = M e^{tJ} M^{-1} Y_0 =$$

$$\frac{1}{2} M \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{t(-1+i)} & 0 \\ 0 & 0 & e^{t(-1-i)} \end{bmatrix} \begin{bmatrix} 4 & -2 & 2 \\ -1 & 1 & -i-1 \\ -1 & 1 & i-1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} =$$

$$\frac{1}{2} M \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{t(-1+i)} & 0 \\ 0 & 0 & e^{t(-1-i)} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = M \begin{bmatrix} e^t \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2+i & 2-i \\ 0 & i & -i \end{bmatrix} \begin{bmatrix} e^t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} e^t \\ e^t \\ 0 \end{bmatrix}$$

$x(t) = e^t$, $y(t) = e^t$, $z(t) = 0$ è la soluzione.

controllo TORNA (vedi sopra)

PROVIAMO CON $Y_0 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ Come sopra

$$Y(t) = \frac{1}{2} M \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{t(-1+i)} & 0 \\ 0 & 0 & e^{t(-1-i)} \end{bmatrix} \begin{bmatrix} 4 & -2 & 2 \\ -1 & 1 & -i-1 \\ -1 & 1 & i-1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\frac{1}{2} M \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{t(-1+i)} & 0 \\ 0 & 0 & e^{t(-1-i)} \end{bmatrix} \begin{bmatrix} 0 \\ -i \\ i \end{bmatrix} =$$

$$\frac{1}{2} M \begin{bmatrix} 0 \\ -i\, e^{t(-1+i)} \\ i\, e^{t(-1-i)} \end{bmatrix} = \frac{i e^{-t}}{2} M \begin{bmatrix} 0 \\ -e^{ti} \\ e^{-ti} \end{bmatrix} =$$

$$\frac{i}{2} e^{-t} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2+i & 2-i \\ 0 & i & -i \end{bmatrix} \begin{bmatrix} 0 \\ -e^{ti} \\ e^{-ti} \end{bmatrix} =$$

$$\frac{i}{2} e^{-t} \begin{bmatrix} -e^{ti} + e^{-ti} \\ -(2+i) e^{ti} + (2-i) e^{-ti} \\ -i\, e^{ti} - i\, e^{-ti} \end{bmatrix} = \frac{-1}{2i} e^{-t} \begin{bmatrix} -e^{ti} + e^{-ti} \\ 2(-e^{ti} + e^{-ti}) - i(e^{ti} + e^{-ti}) \\ -i\,(e^{ti} + e^{-ti}) \end{bmatrix}$$

$$= e^{-t} \begin{bmatrix} \dfrac{e^{ti} - e^{-ti}}{2i} \\ \dfrac{2(e^{ti} - e^{-ti})}{2i} + \dfrac{(e^{ti} + e^{-ti})}{2} \\ \dfrac{(e^{ti} + e^{-ti})}{2} \end{bmatrix} = e^{-t} \begin{bmatrix} \sin(t) \\ 2\sin(t) + \cos(t) \\ \cos(t) \end{bmatrix}$$

$$x(t) = e^{-t} \sin(t)$$
$$y(t) = \left(2\sin(t) + \cos(t)\right) e^{-t}$$
$$z(t) = e^{-t} \cos(t)$$

<u>VERIFICA</u>

$$\begin{cases} x' = 3x - 2y + 3z \\ y' = 5x - 4y + 5z \\ z' = x - y \end{cases}$$

$$x' = -e^{-t}\sin(t) + e^{-t}\cos(t) = e^{-t}\left(-\sin(t) + \cos(t)\right)$$ ok

$$y' = e^{-t}\left(-2\sin(t) - \cos(t) + 2\cos(t) - \sin(t)\right) = e^{-t}\left(-3\sin(t) + \cos(t)\right)$$

ok

$$z' = e^{-t}\left(-\cos(t) - \sin(t)\right)$$

$$3x - 2y + 3z = e^{-t}\left(3\sin(t) - 4\sin(t) - 2\cos(t) + 3\cos t\right) = e^{-t}\left(-\sin(t) + \cos(t)\right)$$

$$5x - 4y + 5z = e^{-t}\left(5\sin(t) - 8\sin(t) - 4\cos(t) + 5\cos(t)\right) = e^{-t}\left(-3\sin(t) + \cos(t)\right)$$

$$x - y = e^{-t}\left(\sin(t) - 2\sin(t) - \cos(t)\right) = e^{-t}\left(-\sin(t) - \cos(t)\right)$$ ok

Cerchiamo la sol della non omogenea con

$$Y_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Potrei usare la formula $Y_1(t) = e^{tA}\left(0 + \int_0^t e^{-\tau A} B(\tau)\, d\tau\right)$

SE PERO' TROVO UNA sol. della non omogenea forse trovo un altro modo

Dato che $B(t) = e^{-t}\begin{pmatrix} 7 \\ 15 \\ 1 \end{pmatrix}$ TENTO di TROVARE

$\bar{Y}(t) = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} e^{-t}$ se metto questo Y_1 nell'equazione trovo

$$-\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} e^{-t} = Y' = e^{-t} A \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} + \begin{pmatrix} 7 \\ 15 \\ 1 \end{pmatrix} e^{-t} \iff$$

$$(A+I)\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} + \begin{pmatrix} 7 \\ 15 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad (A+I)\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} -7 \\ -15 \\ -1 \end{pmatrix}$$

det $\neq 0$ POSSO RISOLVERE

$$\begin{bmatrix} 4 & -2 & 3 \\ 5 & -3 & 5 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} -7 \\ -15 \\ -1 \end{bmatrix}$$

$$\begin{cases} 4\alpha - 2\beta + 3\gamma = -7 \\ 5\alpha - 3\beta + 5\gamma = -15 \\ \alpha - \beta + \gamma = -1 \end{cases} \qquad \begin{cases} 0 \quad (-2+4)\beta + (3-4)\gamma = -7+4 \\ 0 \quad (-3+5)\beta \quad (+5-5)\gamma = -15+5 \\ \alpha - \beta + \gamma = -1 \end{cases}$$

$$\begin{cases} 2\beta - \gamma = 3 \\ 2\beta = -10 \\ \alpha - \beta + \gamma = -1 \end{cases} \qquad \beta = -5 \qquad \gamma = 2\beta - 3 = -10 - 3 \quad \gamma = -13$$

$$\alpha = -1 + \beta - \gamma = -1 - 5 + 13 = 7$$

$\alpha = 7$
$\beta = -5$
$\gamma = -13$

$$\overline{Y}(t)=\begin{pmatrix}7\\-5\\-13\end{pmatrix}e^{-t}$$

VOGLIO LA SOL. Y tale che $Y(0)=\begin{pmatrix}0\\0\\0\end{pmatrix}$.

cerco $Y(t)=\overline{Y}(t)+Y_0(t)$ $\Rightarrow$ $\begin{pmatrix}0\\0\\0\end{pmatrix}=\overline{Y}(0)+Y_0(0)$ $\Rightarrow$

$$Y_0(0)=-\overline{Y}(0)=\begin{pmatrix}-7\\5\\13\end{pmatrix}$$

$$Y_0(t)=e^{tA}\begin{pmatrix}-7\\5\\13\end{pmatrix}=$$

$$\frac{1}{2}M\begin{bmatrix}e^{t}&0&0\\0&e^{t(-1+i)}&0\\0&0&e^{t(-1-i)}\end{bmatrix}\begin{bmatrix}4&-2&2\\-1&1&-i-1\\-1&1&i-1\end{bmatrix}\begin{bmatrix}-7\\5\\13\end{bmatrix}=$$

$$\frac{1}{2}M\begin{bmatrix}e^{t}&0&0\\0&e^{t(-1+i)}&0\\0&0&e^{t(-1-i)}\end{bmatrix}\begin{bmatrix}-28-10+26\\7+5-13(i+1)\\7+5+13(i-1)\end{bmatrix}$$

$$\frac{1}{2}M\begin{bmatrix}e^{t}&0&0\\0&e^{t(-1+i)}&0\\0&0&e^{t(-1-i)}\end{bmatrix}\begin{bmatrix}-12\\-1-13i\\-1+13i\end{bmatrix}=$$

$$\frac{1}{2}M\begin{bmatrix}-12e^{t}\\(-1-13i)e^{t(-1+i)}\\(-1+13i)e^{t(-1-i)}\end{bmatrix}=\frac{e^{t}}{2}M\begin{bmatrix}-12\\(-1-13i)e^{it}\\(-1+13i)e^{-it}\end{bmatrix}=$$

$$\frac{e^{t}}{2}\begin{bmatrix}1&1&1\\1&2+i&2-i\\0&i&-i\end{bmatrix}\begin{bmatrix}-12\\(-1-13i)e^{it}\\(-1+13i)e^{-it}\end{bmatrix}=$$

$$\frac{e^t}{2}\begin{bmatrix} -12 + \underbrace{(-1+13i)e^{it}} + \underbrace{(-1+13i)e^{-it}} \\ -12 + \underbrace{(2+i)(-1-13i)e^{it}} + \underbrace{(2-i)(-1+13i)e^{-it}} \\ \underbrace{i\,(-1-13i)e^{it}} \quad \underbrace{-i\,(-1+13i)e^{-it}} \end{bmatrix} =$$

CONIUGAT

$$\frac{e^t}{2}\begin{bmatrix} -12 + 2\,\mathrm{Re}\,(-1+13i)(\cos(t)+i\sin(t)) \\ -12 + 2\,\mathrm{Re}\left[(2+i)(-1-13i)(\cos(t)+i\sin(t))\right] \\ 2\,\mathrm{Re}\left[i\,(-1-13i)\cdot(\cos(t)+i\sin(t))\right] \end{bmatrix}$$

$$\frac{e^t}{2}\begin{bmatrix} -12 + 2(-\cos(t) - 13\sin(t)) \\ -12 + 2\,\mathrm{Re}\,(11-15i)(\cos(t)+i\sin(t)) \\ 2\,\mathrm{Re}\,(13-i)(\cos(t)+i\sin(t)) \end{bmatrix} =$$

$$\left(\begin{matrix} -(2+i)(1+13i) = \\ -(2-13+15i) \end{matrix}\right)$$

$$e^t\begin{bmatrix} -6 - \cos(t) - 13\sin(t) \\ -6 + 11\cos(t) + 15\sin(t) \\ 13\cos(t) + \sin(t) \end{bmatrix}$$

Se tutto è giusto la sol. Y è data da $\bar{Y} + Y_0$ cioè

$$Y(t) = \begin{bmatrix} 7e^{-t} - 6e^t - (\cos(t) + 13\sin(t))e^t \\ -5e^{-t} - 6e^t + (11\cos(t) + 15\sin(t))e^t \\ -13e^{-t} + (13\cos(t) + \sin(t))e^t \end{bmatrix}$$

THAT'S IT !!

ORALI (PREAPPELLO) 6/6 AULA ?!