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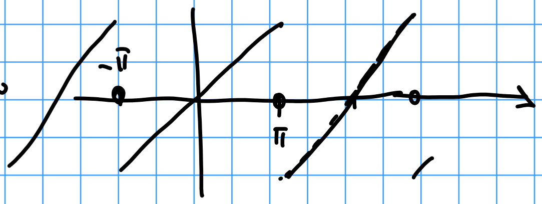
Ricevimento su appuntamento da concordare per email

Altri esempi. Decidiamo che $T = 2\pi$, così $\underline{\omega = 1}$

Esempio



$f(t) = t$ se $-\pi < t < \pi$

$f(-\pi) = f(\pi) = 0$

(e poi estesa a tutto $\mathbb{R}$ in modo 2π-periodico)

Calcoliamo i coeff. di F. REALI

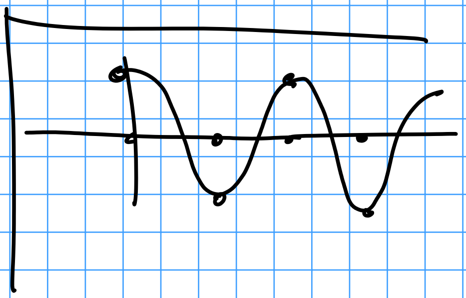
Noto che f è dispari $\Rightarrow$ $a_n = 0$ $\forall n$. Invece

$$b_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(t)\sin(\underset{\overset{\uparrow}{\omega=1}}{n}t)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{t\sin(nt)}_{\text{PARI}}\,dt \quad \left(= \frac{2}{\pi}\int_0^{\pi} t\sin(nt)\,dt\right)$$

(per parti)

$$= \frac{1}{\pi}\left[\underbrace{t\,\frac{\cos(nt)}{-n}}_{\text{dispari}}\right]_{-\pi}^{\pi} + \underbrace{\frac{1}{n\pi}\int_{-\pi}^{\pi}\cos(nt)\,dt}_{=0} =$$

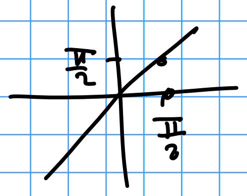
$$\frac{2}{\pi}\,\pi\,\frac{\cos(n\pi)}{-n} = \boxed{-\frac{2(-1)^n}{n}} \leftarrow b_n$$



Quindi (dato f è C^1 a tratti) posso scrivere

$$f(t) = -\sum_{n=1}^{\infty} \frac{2(-1)^n}{n}\sin(nt) \qquad \forall t \in \mathbb{R}$$ (perché ho messo $f(\pi+2k\pi)=0$)

Vediamo cosa ci dà l'eguaglianza sopra se $t = \frac{\pi}{2}$



$$\frac{\pi}{2} = f(\pi/2) = -\sum_{n=1}^{\infty} \frac{2(-1)^n}{n}\sin\left(n\frac{\pi}{2}\right) = (*)$$

$$\sin(n\pi/2) = 0, 1, 0, -1, 0, 1 \ldots = \begin{cases} 0 & \text{se } n = 2k \text{ (pari)} \\ (-1)^k & \text{se } n = 2k+1 \text{ (dispari)} \end{cases}$$

$$(*) = -\sum_{k=0}^{\infty} \frac{2\overset{-1}{\boxed{(-1)^{2k+1}}}}{2k+1}(-1)^k = +\sum_{k=0}^{\infty} \frac{2(-1)^k}{2k+1}$$

ALLA FINE (RI)TROVO

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}$$

$$\left((-1)^{2k+1} = (-1)^{2k}(-1)^1 = \left((-1)^2\right)^k(-1) = (1)^k(-1) = -1 \right)$$

Per curiosità troviamo i c_k

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-int} =$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} t\, e^{-int}\, dt \overset{\text{(per parti)}}{=} \frac{1}{2\pi}\left[t\,\frac{e^{-int}}{-in}\right]_{-\pi}^{\pi} + \frac{1}{2\pi n i}\underbrace{\int_{-\pi}^{\pi} e^{-int}\, dt}_{=0}$$

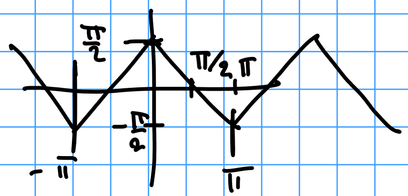
$$= \frac{-1}{2\pi n i}\left(\pi e^{-in\pi} - (-\pi)e^{in\pi}\right) = \frac{-1}{2ni}\left(\underbrace{e^{in\pi} + e^{-in\pi}}_{2\cos(n\pi)}\right) =$$

$$-\frac{1}{ni}\cos(n\pi) = \frac{i\cos(n\pi)}{n} = \frac{i(-1)^n}{n}$$

torna che $b_n = -2\,\mathrm{Im}(c_n) = -\frac{2(-1)^n}{n}$

Anche in questo esempio la serie <u>non</u> converge uniformemente perché f è discontinua. (TORNA COL FATTO che $\sum |a_n| = +\infty$)

<u>Esempio</u>



$$f(t) = \frac{\pi}{2} - |t| \quad \text{se} \quad |t| \leq \pi$$

(e periodicizzata di periodo 2π)

NOTO che f è <u>pari</u> $\Rightarrow b_n = 0$.

posso togliere il modulo

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{f(t)dt}_{\text{PARI}} = \frac{2}{2\pi}\int_0^{\pi}\left(\frac{\pi}{2} - t\right)dt = \frac{1}{\pi}\left[\frac{\pi}{2}t - \frac{t^2}{2}\right]_0^{\pi} = \frac{1}{\pi}\left(\frac{\pi^2}{2} - \frac{\pi^2}{2}\right) = 0$$

$$n \geq 1 \qquad a_n = \frac{\cancel{2}}{\cancel{2}\pi}\int_{-\pi}^{\pi} \underbrace{f(t)\cos(nt)}_{\text{PARI}}dt = \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{2} - t\right)\cos(nt)\,dt =$$

$$(\text{per parti}) = \underbrace{\frac{2}{\pi}\left[\left(\frac{\pi}{2} - t\right)\frac{\sin(nt)}{n}\right]_0^{\pi}}_{\text{FA ZERO a causa del } \sin(n\pi)=0} - \frac{2}{n\pi}\int_0 (-1)\sin(nt)\,dt =$$

$$= \frac{2}{n\pi}\left[\frac{-\cos(nt)}{n}\right]_0^{\pi} = \boxed{\frac{2}{\pi n^2}\left(1 - (-1)^n\right) = a_n} \qquad (n \geq 1)$$

(se calcolassi i c_n troverei $c_n = \frac{1}{\pi n^2}(1 - (-1)^n) \ldots$)

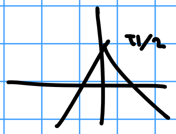
Notiamo che $\sum |a_n| < +\infty$ $\left(|a_n| \leq \frac{4}{\pi}\frac{1}{n^2} \qquad \sum \frac{1}{n^2} < +\infty \right)$

Dunque la serie di Fourier <u>CONVERGE UNIFORMENTE</u>

$$f(t) \underset{\text{UNIF.}}{=} \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(1 - (-1)^n)}{n^2}\cos(nt) = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\cos((2k+1)t)}{(2k+1)^2}$$

↗ ho messo $n = 2k+1$ ($n = 2k$ sparisce)

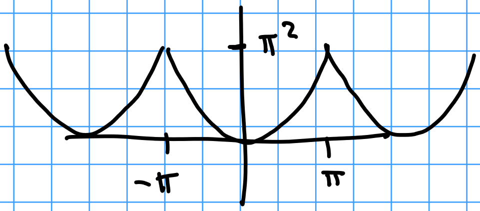
Mettiamo $t = 0$ $\quad f(0) = \frac{\pi}{2}$



Dunque

$$\frac{\pi}{2} = \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \iff \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$

ESEMPIO



$f(t) = t^2$ se $-\pi \leq t \leq \pi$ (periodica di periodo 2π)

Cerchiamo: $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-in(t)}$

$n=0$ $\quad c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2\, dt = \frac{1}{2\pi}\left[\frac{t^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\,\frac{\pi^3}{3} = \frac{\pi^2}{3}$

$n \geq 1$ $\quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2 e^{-int}\, dt = \frac{1}{2\pi}\underbrace{\left[\frac{t^2 e^{-int}}{-in}\right]_{-\pi}^{\pi}}_{\text{FA ZERO PERCHÉ } e^{-int} \text{ è } 2\pi\text{-periodica e } (-\pi)^2=(\pi)^2} + \frac{1}{\cancel{2}\pi n i}\int_{-\pi}^{\pi} \cancel{2} t e^{-int}\, dt$

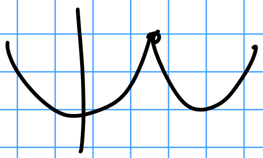
$$\frac{1}{\pi n i}\left[\frac{t e^{-int}}{-in}\right]_{-\pi}^{\pi} + \frac{1}{\pi n^2 i(-i)}\underbrace{\int_{-\pi}^{\pi} e^{-int}\, dt}_{=0} = \frac{1}{\pi n^2}\left(\pi e^{-in\pi} + \pi e^{in\pi}\right) =$$

$i(-i) = 1$ $\qquad \frac{2}{n^2}\cos(n\pi)$

$b_n = 0$ $\quad \boxed{a_n = \frac{4}{n^2}(-1)^n}$ $\quad \boxed{a_0 = \frac{\pi^2}{3}}$

$c_n = \frac{2}{n^2}(-1)^n$ $\qquad \left(\sum |c_n| < +\infty \;! \Rightarrow \text{CONV. UNIF.}\right)$

Mettiamo $t = \pi$



$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos(n\pi) = \frac{\pi^2}{3} + \sum_{1}^{\infty} \frac{4}{n^2} \iff$$

$$\sum_{1}^{\infty} \frac{1}{n^2} = \frac{1}{4}\left(\pi^2 - \frac{\pi^2}{3}\right) = \frac{1}{4}\left(\frac{2}{3}\pi^2\right) = \frac{\pi^2}{6}$$

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