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Lezioni di Analisi Matematica 2 e Complementi

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ESERCIZIO calcolare

$$\iiint_B xyz \, dx\,dy\,dz \qquad B = \{x^2+y^2+z^2 \le 1,\ x \ge 0\ y \ge 0\ z \ge 0\}$$

I° modo: scrivo B come insieme "normale" e procedere e integrazioni iterate

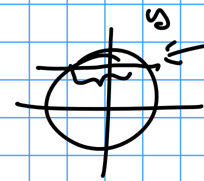
$\{x^2+y^2 \le R^2\ \ x \ge 0,\ y \ge 0\}$
$\{0 \le y \le R,\ \ 0 \le x \le \sqrt{R^2-y^2}\}$

$$B = \{0 \le z \le 1,\ \ x^2+y^2 \le 1-z^2,\ x \ge 0,\ y \ge 0\} =$$
$$\{0 \le z \le 1,\ \ \underbrace{0 \le y \le \sqrt{1-z^2}},\ \ \underbrace{0 \le x \le \sqrt{1-z^2-y^2}}\}$$

z

$$\Rightarrow \text{INTEGRALE} = \int_0^1 dz \int_0^{\sqrt{1-z^2}} dy \int_0^{\sqrt{1-y^2-z^2}} dx \ \ xyz =$$

$$\int_0^1 z \left(\int_0^{\sqrt{1-z^2}} y \left(\int_0^{\sqrt{1-y^2-z^2}} x \ dx \right) dy \right) dz =$$



voglio $\sqrt{1-z^2}$

$$\int_0^1 z \left(\int_0^{\sqrt{1-z^2}} y \ \ \frac{1}{2}(1-y^2-z^2)\, dy \right) dz = \qquad \left(\int x = \frac{x^2}{2} + c \right)$$

$$\frac{1}{2} \int_0^1 z \, dz \int_0^{\sqrt{1-z^2}} (y - y^3 - z^2 y)\, dy = \frac{1}{2} \int_0^1 z \left[\frac{y^2}{2} - \frac{y^4}{4} - \frac{z^2 y^2}{2} \right]_0^{\sqrt{1-z^2}} dz =$$

$$\frac{1}{8}\int_0^1 z\left(2(1-z^2)-(1-z^2)^2-2z^2(1-z^2)\right)dz =$$

$$\frac{1}{8}\int_0^1 z\left(2-2z^2-(1-2z^2+z^4)-2z^2+2z^4\right)dz =$$

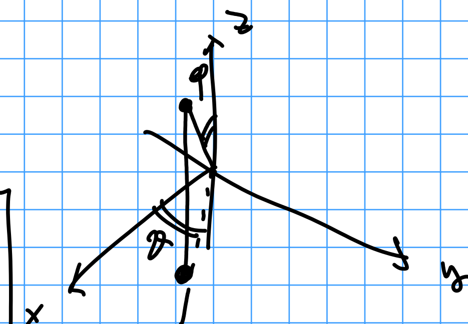
$$\frac{1}{2\cdot 8}\int_0^1 2z\left(1-2z^2+z^4\right)dz = \qquad s=z^2 \qquad ds = 2z\,dz$$

$$\frac{1}{16}\int_0^1 (1-2s+s^2)\,ds = \frac{1}{16}\left[s-s^2+\frac{s^3}{3}\right]_0^1 = \frac{1}{16}\left(1-1+\frac{1}{3}\right) = \frac{1}{48}$$

II° (IN COORDINATE SFERICHE) $\quad \rho \geq 0,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \varphi \leq \pi$

$$x = \rho\cos\theta\sin\varphi \qquad y = \rho\sin\theta\sin\varphi \qquad z = \rho\cos\varphi$$

$$\Phi(\rho, \theta, \varphi)$$



MI SERVE $|\det J_\Phi|$ (per il cambio di variabile)

$$J_\Phi = \begin{bmatrix} \cos\theta\sin\varphi & -\rho\sin\theta\sin\varphi & \rho\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & \rho\cos\theta\sin\varphi & \rho\sin\theta\cos\varphi \\ \cos\varphi & 0 & -\rho\sin\varphi \end{bmatrix}$$

$$\det J_\Phi = \rho^2\sin\varphi \det\begin{bmatrix} \cos\theta\sin\varphi & -\sin\theta & \cos\theta\cos\varphi \\ \sin\theta\sin\varphi & \cos\theta & \sin\theta\cos\varphi \\ \cos\varphi & 0 & -\sin\varphi \end{bmatrix} =$$

$$\rho^2\sin\varphi\left(\cos\varphi\left(-\sin^2\theta\cos\varphi-\cos^2\theta\cos\varphi\right)-\sin\varphi\left(\cos^2\theta\sin\varphi+\sin^2\theta\sin\varphi\right)\right)$$

$$\rho^2\sin\varphi\left(-\cos^2\varphi-\sin^2\varphi\right) = -\rho^2\sin\varphi$$

$$|\det J_\Phi| = \rho^2\sin\varphi \qquad (\text{perché } \sin\varphi \geq 0 \quad 0 \leq \varphi \leq \pi)$$

$$\iiint_B xyz\,dx\,dy\,dz \rightsquigarrow \text{COORD. SFERICHE} \qquad \boxed{\begin{array}{l} 0 \leq \theta \leq \pi/2 \\ 0 \leq \varphi \leq \pi/2 \\ 0 \leq \rho \leq 1 \end{array}}$$

$$= \iiint_{\{0\leq\rho\leq1,\ 0\leq\theta\leq\pi/2,\ 0\leq\varphi\leq\pi/2\}} (\rho\cos\theta\sin\varphi)(\rho\sin\theta\sin\varphi)(\rho\cos\varphi)\,\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$$

$$= \left(\underbrace{\int_0^1 \rho^5\, d\rho}_{1/6}\right)\left(\int_0^{\pi/2} \cos\theta \sin\theta\, d\theta\right)\left(\int_0^{\pi/2} \sin^3\varphi \cos\varphi\, d\varphi\right)$$

($x \geq 0 \quad y \geq 0 \quad z \geq 0 \longrightarrow \cos\varphi \geq 0 \iff 0 \leq \varphi \leq \pi/2$;

$\rho\cos\theta \geq 0 \quad \rho \sin\varphi \geq 0 \iff 0 \leq \theta \leq \pi/2$)

$$\frac{1}{6} \int_0^{\pi/2} \frac{1}{2} \sin 2\theta \, d\theta \int_0^1 s^3\, ds$$

$\sin\varphi = s$

$\cos\varphi\, d\varphi = ds$

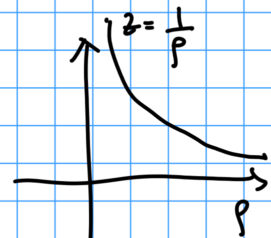
$$\frac{1}{6}\,\frac{1}{2}\underbrace{\left[\frac{-\cos 2\theta}{2}\right]_0^{\pi/2}}_{1}\left[\frac{s^4}{4}\right]_0^1 = \frac{1}{6\cdot 8} = \frac{1}{48}$$

TORNA

ESERCIZIO

Dato $\alpha > 0$ considero

$$\iiint_D \underbrace{\frac{z}{(x^2+y^2)^\alpha}}_{f(x,y,z)}\, dx\,dy\,dz \qquad D = \left\{ 0 \leq z \leq 1,\ x^2+y^2 \leq \frac{1}{z^2} \right\}$$



← RUOTO ATTORNO A z

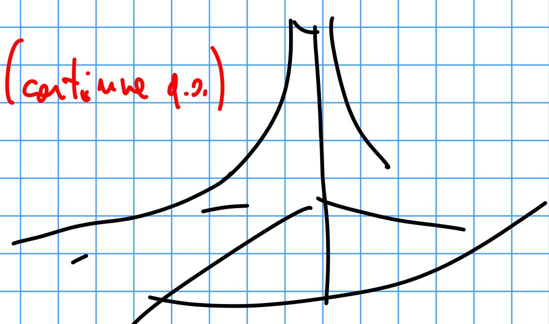
f è continua su $\mathbb{R}^3 \setminus \{x=y=0\}$

MA $\{x=y=0\}$ è trascurabile in $\mathbb{R}^3$

$\Rightarrow$ f è misurabile

D è dato da funzioni misurabili (continue q.o.)

$\Rightarrow$ D è misurabile



L'INTEGRALE HA SENSO

MA POTREBBE ESSERE $+\infty$

CALCOLIAMO L'INTEGRALE (al variare di $\alpha > 0$)

$$\iiint_D f = \int_0^1 z\, dz \iint_{\left\{x^2+y^2 \leq \frac{1}{z^2}\right\}} \frac{dx\,dy}{(x^2+y^2)^\alpha} =$$ (coord. polari in xy)

$$\int_0^1 z\,dz \int_0^{2\pi} d\theta \int_0^{1/z} \frac{\rho\, d\rho}{\rho^{2\alpha}} = 2\pi \int_0^1 z\,dz \int_0^{1/z} \rho^{1-2\alpha}\, d\rho$$

$$2\pi \int_0^1 z\,dz \left[\frac{\rho^{2-2\alpha}}{2-2\alpha} \right]_0^{1/z}$$ se $2-2\alpha \neq 0 \iff \boxed{\alpha \neq 1.}$

$(\alpha = 1$ dopo!$)$

$$= \begin{cases} +\infty & \text{se } 2-2\alpha < 0 \iff \alpha > 1 \\ \dfrac{\pi}{1-\alpha} \displaystyle\int_0^1 z\, z^{-(2-2\alpha)}\, dz = & \boxed{0 < \alpha < 1} \end{cases}$$

$$\frac{\pi}{1-\alpha} \int_0^1 z^{1-2+2\alpha}\, dz = \frac{\pi}{1-\alpha}\left[\frac{z^{2\alpha}}{2\alpha}\right]_0^1 = \frac{\pi}{2(1-\alpha)\alpha}$$

Se $\alpha = 1$ TROVO $\underbrace{2\pi \int_0^1 z\, dz \int_0^{1/z} \frac{d\rho}{\rho}}_{= +\infty}$ $\left(\left[\ln \rho \right]_0^{1/z} = +\infty \right)$

IN DEFINITIVA l'integrale $= \begin{cases} +\infty & \text{se } \alpha \geq 1 \\ \dfrac{\pi}{2(1-\alpha)\alpha} & \text{se } 0 < \alpha < 1 \end{cases}$

INTEGRALE NOTEVOLE $\displaystyle\int_{-\infty}^{+\infty} e^{-x^2}\, dx = (*)$

(si sa che non è possibile esprimere la primitiva di e^{-x^2} in termine di funzioni elementari). PERÒ CALCOLO $(*)$ TRUCCO!

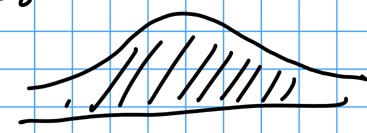
$$(*)^2 = \int_{-\infty}^{+\infty} e^{-x^2} dx \int_{-\infty}^{+\infty} e^{-y^2} dy \underset{(\text{TONELLI})}{=} \iint_{\mathbb{R}^2} e^{-x^2} e^{-y^2}\, dx\, dy =$$

$$\iint_{\mathbb{R}^2} e^{-x^2-y^2}\, dx\, dy = \text{coord. polari}$$

$$\iint_{\{0 \leq \theta \leq 2\pi \ \ \rho \geq 0\}} e^{-\rho^2} \rho\, d\rho\, d\theta \underset{(\text{TONELLI})}{=} \int_0^{2\pi} d\theta \int_0^{+\infty} \rho e^{-\rho^2} d\rho = \pi \int_0^{+\infty} 2\rho\, e^{-\rho^2} d\rho$$

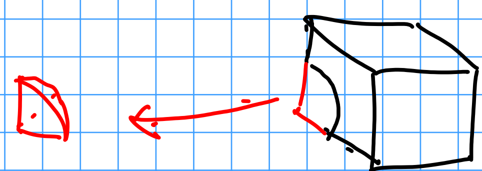
$$(s=\rho^2) = \pi\int_0^{+\infty} e^{-s}\,ds = \pi\left[-e^{-s}\right]_0^{+\infty} \quad \pi$$

$$\Rightarrow \int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}$$



ESERCIZIO

$$\iiint_D \frac{xy}{\sqrt{z}}\,dx\,dy\,dz \qquad D=\{x^2+y^2+z^2\ge 1,\ 0\le x\le 2,\ 0\le y\le 2,\ 0\le z\le 2\}$$



NOTA $D = Q\setminus B$ $\qquad Q=\{0\le x\le 2,\ 0\le y\le 2,\ 0\le z\le 2\}$

$B=\{x^2+y^2+z^2<1\}$

$$\rightsquigarrow \iiint_D \sim = \iiint_Q - \iiint_B$$

NO perchè $B \subset Q$

NO

$$\iiint_D = \iiint_Q - \iiint_{B\cap Q}$$

← SOLO se trovo valori FINITI

Calcoliamo $\iiint_D$ e $\iiint_{B\cap Q}$ $\quad (B\cap Q=\{x^2+y^2+z^2<1,\ x\ge 0,\ y\ge 0,\ z\ge 0\})$

$$\iiint_Q \frac{xy}{\sqrt{z}}\,dx\,dy\,dz$$

(se venisse $+\infty$ sarei nei guai)

$$\int_0^2 x\,dx\int_0^2 y\,dy\int_0^2\frac{dz}{\sqrt{z}} = \left[\frac{x^2}{2}\right]_0^2\left[\frac{y^2}{2}\right]_0^2\left[2\sqrt{z}\right]_0^2 = 2\cdot 2\cdot 2\sqrt{2} = \boxed{8\sqrt{2}}$$

$$\iiint_{B\cap Q}\frac{xy}{\sqrt{z}}\,dx\,dy\,dz$$

← coordinate sferiche

$0\le\rho<1 \qquad 0\le\theta\le\frac{\pi}{2} \qquad 0\le\varphi\le\frac{\pi}{2}$

$$\int_0^{\pi/2}d\theta\int_0^{\pi/2}d\varphi\int_0^1 \sin\varphi\,\rho^2\,d\rho\ \frac{\rho\cos\theta\sin\varphi\ \rho\sin\theta\sin\varphi}{\sqrt{\rho\cos\varphi}} =$$

$$\int_0^{\pi/2}\frac{1}{2}\sin 2\theta\,d\theta\int_0^{\pi/2}\frac{\sin^3\varphi}{\sqrt{\cos\varphi}}\,d\varphi\int_0^1\rho^{4-1/2}\,d\rho =$$

$$\underbrace{\frac{1}{2}\left[\frac{-\cos 2\theta}{2}\right]_0^{\pi/2}}_{1} \int_0^{\pi/2} \frac{1-\cos^2\varphi}{\sqrt{\cos\varphi}} \sin\varphi \, d\varphi \left[\frac{2}{9}\rho^{9/2}\right]_0^1 \qquad \left(\begin{array}{l} s=\cos\varphi \\ ds=-\sin\varphi\, d\varphi \end{array}\right)$$

$$\frac{1}{2}\int_1^0 \frac{1-s^2}{\sqrt{s}}(-ds) \cdot \frac{2}{9} = \frac{1}{9}\int_0^1 \frac{1-s^2}{\sqrt{s}}\, ds = \frac{1}{9}\left[2s^{1/2}-\frac{2}{5}s^{5/2}\right]_0^1 =$$

$$\frac{1}{9}\left(2-\frac{2}{5}\right) = \boxed{\frac{8}{45}}$$

INTEGRALE INIZIALE = $\boxed{8\sqrt{2} - \frac{8}{45}}$

PASSAGGI AL LIMITE SOTTO IL SEGNO DI INTEGRALE

PROBLEMA Ho una "successione di funzioni integrabili"

- $\forall n$ è data $f_n(x)$ integrabile su $\mathbb{R}^N$ $\left(f_n : \mathbb{R}^N \to [-\infty, +\infty]\right)$
- So che $\overset{\forall x}{\vee}$ esiste $\lim_{n\to\infty} f_n(x) = f(x)$

In realtà basta che per quasi ogni x esista il limite -

IN QUESTO CASO $f(x)$ è definito per quasi ogni x.

PERÒ $\int_{\mathbb{R}^N} f\, dx$ HA SENSO perché in qualunque modo

io estenda f l'integrale non cambia

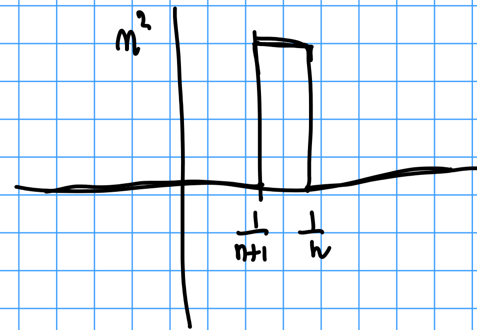
MI CHIEDO SE (a) f è integrabile

(b) $\int_{\mathbb{R}^N} f(x)\,dx = \lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\,dx$

(b) $\Leftrightarrow$ $\lim_{n\to\infty} \int_{\mathbb{R}^N} f_n(x)\,dx = \int_{\mathbb{R}^N} \lim_{n\to\infty} f_n(x)\,dx$

IN GENERALE LA RISPOSTA È $\boxed{\text{NO}}$

CONTROESEMPIO $\quad f_m : \mathbb{R} \to \mathbb{R}$ definito da



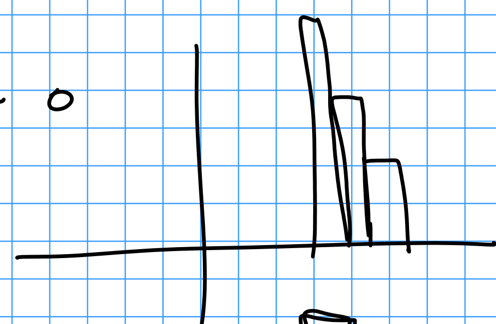
$$f_m(x) = \begin{cases} 0 & \text{se } x > 1/m \\ m^2 & \\ 0 & \text{se } x < \frac{1}{m+1} \end{cases}$$

$$f_m = m^2 \mathbb{1}_{\left[\frac{1}{n+1}, \frac{1}{n}\right]}$$

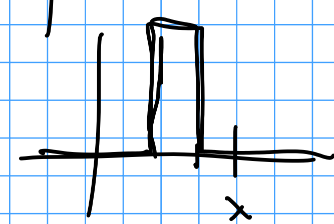
$$\left(\mathbb{1}_A(x) = \begin{cases} 0 & \text{se } x \notin A \\ 1 & \text{se } x \in A \end{cases} \right)$$

Fisso $x \in \mathbb{R}$ $\quad \lim_{m \to \infty} f_m(x)$

se $x \leq 0$ $\quad f_m(x) = 0 \Rightarrow$ limite $= 0$



se $x > 0$ $\quad$ per $n > \frac{1}{x} \Rightarrow \quad x > \frac{1}{n}$



DUNQUE $\quad$ se $x > 0$, $\quad f_n(x) = 0 \quad \forall\, m > \frac{1}{x}$

$\Rightarrow$ quindi per $x > 0$ $\quad \lim_{n \to \infty} f_m(x) = 0$

DUNQUE $f(x) = 0$

SARÀ VERO che $\quad \lim_{m \to \infty} \int_{-\infty}^{+\infty} f_m(x)\,dx = \int_{-\infty}^{+\infty} 0 \, dx = 0$

||

CALCOLIAMO $\quad \int_{\frac{1}{n+1}}^{\frac{1}{n}} n^2 \, dx = m^2 \left(\frac{1}{n} - \frac{1}{n+1} \right) = \frac{n^2}{n(n+1)} \to 1$

NON VALE !!

TEOREMA DI LEBESGUE (della convergenza dominata)

SUPPONIAMO

- $f_n : \mathbb{R}^N \to [-\infty, +\infty]$ misurabili
- $g : \mathbb{R}^N \to [0, +\infty]$ INTEGRABILE
- PER q.o. $x \in \mathbb{R}^N$ esiste

$$\lim_{n \to \infty} f_n(x) =: f(x) \qquad \left(f \text{ è definita quasi ovunque} \right)$$

- $\forall x \in \mathbb{R}^N \qquad |f_n(x)| \le g(x)$ $\qquad$ g DOMINA le $|f_n|$

$\Rightarrow$ f_n sono tutte integrabili

$\Rightarrow$ f è integrabile e vale

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} f_n(x)\,dx = \int_{\mathbb{R}^N} f(x)\,dx$$

(NO DIM.)

nell'esempio di prima "l'inviluppo delle f_n"

$\leadsto$ $g(x) = \frac{1}{x}$ che non è integrabile