Claudio Saccon (*)
Ingegneria Aerospaziale

Lezioni di Analisi Matematica 2 e Complementi

Lezione 71 26/05/21

email: claudio.sacconCHIOCCIOLAunipi.it
web: http://pagine.dm.unipi.it/csblog1/

Ricevimento su appuntamento da concordare per email

$$\begin{cases} x' = -2x - 4y + 4z \\ y' = 3y - z \\ z' = y + z \end{cases} \qquad \begin{aligned} x(0) &= 1 \\ y(0) &= 0 \\ z(0) &= 1 \end{aligned} \qquad Y_0 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} -2 & -4 & 4 \\ 0 & 3 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$ devo trovare e^{tA};

$$p(\lambda) = \det \begin{bmatrix} -2-\lambda & -4 & 4 \\ 0 & 3-\lambda & -1 \\ 0 & 1 & 1-\lambda \end{bmatrix} = -(2+\lambda) \det \begin{bmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix} =$$

$$-(2+\lambda)\left((3-\lambda)(1-\lambda)+1\right) = -(2+\lambda)\left(4 - 4\lambda + \lambda^2\right) = -(2+\lambda)(\lambda-2)^2$$

autovalori: $\lambda_1 = -2$ $m_A = 1$, $\lambda_2 = 2$ $m_A = 2$

- $\lambda = -2$ c'è un solo blocco $J_1(2) = [2]$
- $\lambda = 2$ considero $B = A - 2I = \begin{bmatrix} -4 & -4 & +4 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$ ← HA RANGO 2

$\Rightarrow m_G = \dim \operatorname{Ker} B = 1$

So che B^2 deve avere nucleo di dim 1. Però devo calcolare B^2

$$\begin{bmatrix} -4 & -4 & 4 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} -4 & -4 & 4 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 16 & 16 & -16 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Mi serve un elemento $\underbrace{e_3 \in \ker B^2 \setminus \ker B}$. Se $e_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$z = x + y$

$$\Rightarrow \begin{bmatrix} x \\ y \\ x+y \end{bmatrix}$$

Mi serve che $Be_3 \neq 0$

Prendo e metto $y=0$ $x=1 \Rightarrow z$ $(e_3 = Y_0)$ Infatti

$$Be_3 = \begin{bmatrix} -4 & -4 & 4 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} = e_2 \neq 0$$

Ho trovato la sequenza per λ_2 $\quad e_3 \xrightarrow{B} e_2 \xrightarrow{B} 0$

TORNIAMO A $\lambda_1 = -2$. Mi serve e_1 autovettore $e_1 \in \ker(A+2I)$

$$\begin{bmatrix} 0 & -4 & 4 \\ 0 & 5 & -1 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \qquad e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \qquad J = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Risolvo l'equazione

$$Y(t) = e^{tA} Y_0 = M e^{tJ} M^{-1} e_3 = M e^{tJ} \hat{e}_3 =$$

($M^{-1} e_3 = \hat{e}_3 = e_3$)

$$M \begin{bmatrix} e^{-2t} & 0 & 0 \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ te^{2t} \\ e^{2t} \end{bmatrix} =$$

$$e^{2t}\begin{bmatrix} 1 \\ -t \\ -t+1 \end{bmatrix}$$

$x(t) = e^{2t}$

$y(t) = -te^{2t}$

$z(t) = -te^{2t} + e^{2t}$

LUNEDÌ ORE 9 - 11 lezione di recupero

FORMULA PER IL CASO NON OMOGENEO

(P.L.) $\quad Y' = AY + B(t)$

Mi serve una soluzione particolare. Lo trovo dalla formula:

$$(\otimes) \qquad \bar{Y}(t) = \int_0^t e^{(t-s)A} B(s)\, ds$$

Infatti se deriva:

$$\bar{Y}'(t) = \left[e^{(t-s)A} B(s)\right]_{s=t} + \int_0^t \frac{d}{dt} e^{(t-s)A} B(s)\, ds =$$

$$\underbrace{e^{0A}}_{I} B(t) + \int_0^t A e^{(t-s)A} B(s)\, ds =$$

$$B(t) + A\int_0^t e^{(t-s)A} B(s)\, ds = B(t) + A\bar{Y}(t)$$

NOTA Se $\bar{Y}$ è definita in $(\otimes)$ $\Rightarrow$ $\bar{Y}(0) = 0$.

ESEMPIO Risolvere

$$\begin{cases} x' = 3x + t \\ y' = y + 1 \\ z' = y + z \end{cases}$$

con dato iniziale $x(0) = 1 \quad y(0) = 0 \quad z(0) = -1$

La matrice è $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ Termine noto $B(t) = \begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix}$

$P(\lambda) = (3-\lambda)(1-\lambda)^2 \Rightarrow$ autov. $\lambda_1 = 3$ $\underbrace{\lambda_2 = 1}_{m_A = 2}$

$\lambda = 3$ autovettore $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda = 1$ $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $(A - I)$ ha rango 2

$\left(B^k = \begin{bmatrix} 2^k & 0 \\ \phi & 1 \\ & 0 \end{bmatrix} \right)$ $\updownarrow$ $\left(B^k \text{ ha rango } 1 \ \forall k \geq 2 \right)$

$B^2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\dim(\ker B^2) = 2 = m_A(1)$

un elemento $e_3 \in \ker B^2 \setminus \ker B$ $(x = 0)$ $e_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$B e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = e_2$

DUNQUE $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

e si vede che $M^{-1} = M$

(MA scambia II e III colonna di A

AM scambia II e III riga di A)

$\Rightarrow A = M J M$ dove $J = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$MJ = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$(MJ)M = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = A$ (TORNA)

$\Rightarrow$

$$e^{tA} = M \begin{bmatrix} e^{3t} & 0 & 0 \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{bmatrix} M = \begin{bmatrix} e^{3t} & 0 & 0 \\ 0 & e^t & 0 \\ 0 & te^t & e^t \end{bmatrix}$$

$$\left(e^{tA} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} e^{3t} \\ 0 \\ -e^t \end{bmatrix} \quad ☆ \right)$$

$$\bar{Y}(t) = \int_0^t e^{(t-s)A} B(s)\,ds$$ mettiamo $t-s=\sigma \quad s = t-\sigma \quad ds = -d\sigma$

$$\int_0^t e^{\sigma A} B(t-\sigma)\,d\sigma =$$

$$\int_0^t \begin{bmatrix} e^{3\sigma} & 0 & 0 \\ 0 & e^\sigma & 0 \\ 0 & \sigma e^\sigma & e^\sigma \end{bmatrix} \begin{bmatrix} t-\sigma \\ 1 \\ 0 \end{bmatrix} d\sigma = \int_0^t \begin{bmatrix} (t-\sigma)e^{3\sigma} \\ e^\sigma \\ \sigma e^\sigma \end{bmatrix} d\sigma$$

$$\bar{X}(t) = \int_0^t (t-\sigma)e^{3\sigma}\,d\sigma = \left[(t-\sigma)\frac{e^{3\sigma}}{3} \right]_{\sigma=0}^{\sigma=t} + \int_0^t \frac{e^{3\sigma}}{3}\,d\sigma =$$

$$-\frac{t}{3} + \frac{e^{3t}}{9} - \frac{1}{9} = \frac{e^{3t} - 3t - 1}{9}$$

$$\bar{Y}(t) = e^t - 1$$

$$\bar{Z}(t) = \int_0^t \sigma e^\sigma\,d\sigma = \left[\sigma e^\sigma \right]_0^t - \int_0^t e^\sigma\,d\sigma =$$

$$t e^t - e^t + 1$$

$$\bar{X}(t) = \frac{e^{3t} - 3t - 1}{9} \qquad \bar{Y}(t) = e^{t-1} \qquad \bar{Z}(t) = te^t - e^t + 1$$

Verifichiamo una delle eq. (la III°) ??

$$z' = e^t + te^t - e^t = te^t$$

$$y + z = \underbrace{te^t - e^t + 1}_{z} + e^t - 1 = te^t \qquad \text{TORNA !}$$

PERO' NON VERIFICA $z(0) = -1$ $\quad x(0) = 1$

DEVO AGGIUNGERE $e^{tA} Y(0)$ (A)

$\vec{g}$ $\vec{F}$

di solito si chiama $\vec{F}$ il pot. vett. cioè:

$$\boxed{\text{rot}\, \vec{F} = \vec{g}}$$

Se uso Stokes, ho, $\iint_S \vec{g} \cdot \hat{\nu}\, d\sigma = \int_{\partial(S)_\nu} \vec{F} \cdot d\vec{s}$