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Lezioni di Analisi Matematica 2 e Complementi

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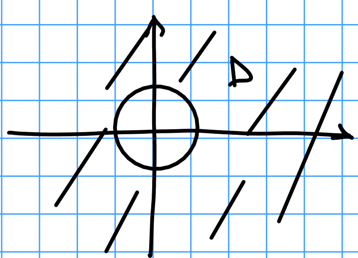
Ricevimento su appuntamento da concordare per email

Domanda Data la funzione definita su $\mathbb{R}^2$ eccetto $(0,0)$
$\mathbb{R}^2 \setminus \{(0,0)\}$

$$f(x,y) = \frac{x}{x^2+y^2}$$

Dire se f ammette integrale su $D = \{(x,y) : x^2+y^2 \geq 1\}$



Nota una funzione f ammette integrale

se
1. è misurabile ed è ≥ 0 (può essere $\int_D f = +\infty$)
2. è integrabile $\Longleftrightarrow$ f è misurabile e su D f^+, f^- hanno integrale finito ($\int_D f^+ < +\infty$ $\int_D f^- < +\infty$) $\Longleftrightarrow$ f è misurabile e $\int_D |f| < +\infty$

- Nel caso in esame f è misurabile poiché f è continua su D (su $\mathbb{R}^2 \setminus \{(0,0)\}$ — $\{(0,0)\}$ ha misura nulla (è trascurabile))
- La nostra f non è ≥ 0. Dunque devo vedere se f rientra nel caso (2). Vediamo con f^+ e f^-

$$(i)\ \int_D f^+ = \int_{\{x^2+y^2\geq 1,\, x\geq 0\}} \frac{x}{x^2+y^2}\,dx\,dy$$

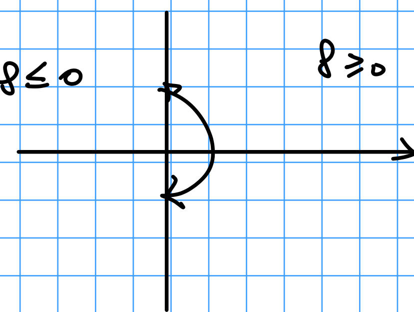
perché f sia integrabile devono essere entrambi finiti

$$(ii)\ \int_D f^- = \int_{\{x^2+y^2\geq 1,\, x\leq 0\}} \frac{-x}{x^2+y^2}\,dx\,dy$$

Uso le coordinate polari $x=\rho\cos\theta$ $y=\rho\sin\theta$ $dxdy=\rho\,d\rho\,d\theta$

$$(i) = \iint_{\{\rho\geq 1,\ -\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}\}} \frac{\rho\cos\theta}{\rho^2}\,\rho\,d\rho\,d\theta =$$

$$= \underbrace{\int_{-\pi/2}^{\pi/2}\cos(\theta)\,d\theta}_{=2}\ \underbrace{\int_1^{+\infty} d\rho}_{=+\infty} = +\infty$$



(RICORDO: Nell'ambito dell'integrazione si conviene che $0\cdot\infty=0$
Con questa convenzione i teoremi sono validi)

$\Rightarrow \int_D f^+ = +\infty \Rightarrow$ l'integrale di f NON ESISTE

(Nello stesso modo trovo $\int_D f^- = +\infty$: $\int_{\frac{\pi}{2}}^{\frac{3}{2}\pi}(-\cos(\theta)) \int_1^{+\infty} d\rho = +\infty$)

Alternativamente avrei potuto calcolare:

$$\int_D |f|\,dx\,dy = \underbrace{\int_0^{2\pi}|\cos(\theta)|\,d\theta}_{=2\int_{\frac{\pi}{2}}^{3/2\pi}-\cos(\theta)\,d\theta=4}\ \int_1^{+\infty} d\rho = +\infty$$

Le cosa da non fare è:

$$\int_D f = \underbrace{\int_0^{2\pi}\cos(\theta)\,d\theta}_{=0}\ \int_1^{+\infty} d\rho = 0$$

DUNQUE: f NON AMMETTE INTEGRALE

ESERCIZIO $\displaystyle\iiint_A \frac{xy}{\sqrt{z}}\,dx\,dy\,dz$ dove:

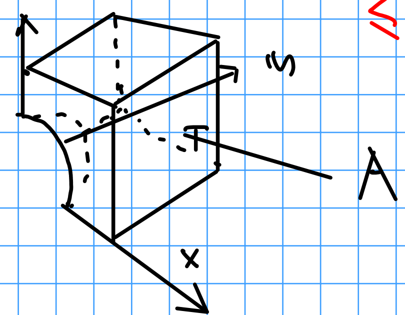
$$A = \{ 0 \le x \le 2,\ 0 \le y \le 2,\ 0 \le z \le 2,\ x^2+y^2+z^2 \geqslant 1 \}$$

OSSERVO CHE $A = Q \setminus B = Q \setminus B^+$

$$Q = \{ 0 \le x \le 2, 0 \le y \le 2, 0 \le z \le 2 \}$$

$$B = \{ x^2+y^2+z^2 < 1 \}$$

$\left(m(\{x^2+y^2+z^2=1\}) = 0 \right)$



$B^+ = B \cap Q$

$\{ x^2+y^2+z^2 < 1,\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0 \}$

Con queste posizioni posso scrivere

$$(*) \quad \iiint_A \frac{xy}{\sqrt{z}}\,dx\,dy\,dz = \iiint_Q \frac{xy}{\sqrt{z}}\,dx\,dy\,dz - \iiint_{B^+} \frac{xy}{\sqrt{z}}\,dx\,dy\,dz$$

$\left(A \cup B^+ = Q \;;\; \text{la formula che uso: } \int_A f + \int_{B^+} f = \int_Q f \right)$

- $$\iiint_Q \frac{xy}{\sqrt{z}} = \left(\int_0^2 x\,dx\right)\left(\int_0^2 y\,dy\right)\left(\int_0^2 \frac{dz}{\sqrt{z}}\right) = \left[\frac{x^2}{2}\right]_0^2 \left[\frac{y^2}{2}\right]_0^2 \left[2z^{1/2}\right]_0^2$$

NOTA che $\frac{1}{\sqrt{z}}$ è integrabile su $[0,2]$

$$= \frac{4}{2} \cdot \frac{4}{2} \cdot 2\sqrt{2} = 8\sqrt{2}$$

- $$\iiint_{B^+} \frac{xy}{\sqrt{z}}\,dx\,dy\,dz = (\text{coordinate sferiche}) =$$

$$x = \rho\cos\theta\sin\psi \qquad y = \rho\sin\theta\sin\psi \qquad z = \rho\cos\psi$$

$$\Downarrow \quad dx\,dy\,dz = \rho^2 \sin\psi \; d\theta\, d\psi\, d\rho$$

$$B^+ \longrightarrow \left\{ 0 \le \theta \le \frac{\pi}{2},\quad 0 \le \psi \le \frac{\pi}{2},\quad 0 \le \rho \le 1 \right\}$$

$$= \int_0^{\frac{\pi}{2}} \left(\int_0^{\frac{\pi}{2}} \left(\int_0^1 \frac{\rho\cos\theta\, \rho\sin\theta\, \sin^2\psi\, \sin\psi}{\sqrt{\rho\cos\psi}}\, \rho^2 d\rho \right) d\psi \right) d\theta =$$

$$\int_0^{\frac{\pi}{2}} \cos\theta \sin\theta\, d\theta \int_0^{\frac{\pi}{2}} \frac{\sin^3\psi}{\sqrt{\cos\psi}}\, d\psi \int_0^1 \rho^{7/2} d\rho =$$

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 2\theta\, d\theta \int_0^{\frac{\pi}{2}} \frac{1-\cos^2(\psi)}{\sqrt{\cos\psi}} \sin\psi\, d\psi \left[\frac{2}{9}\rho^{9/2}\right]_0^1 =$$

$$\frac{1}{2} \underbrace{\left[\frac{-\cos(2\theta)}{2}\right]_0^{\frac{\pi}{2}}}_{=1} \int_1^0 \frac{1-s^2}{\sqrt{s}} \cdot -ds \cdot \frac{2}{9} = \quad \leftarrow$$

$$\cos(\psi) = s \qquad -\sin\psi\, d\psi = ds$$

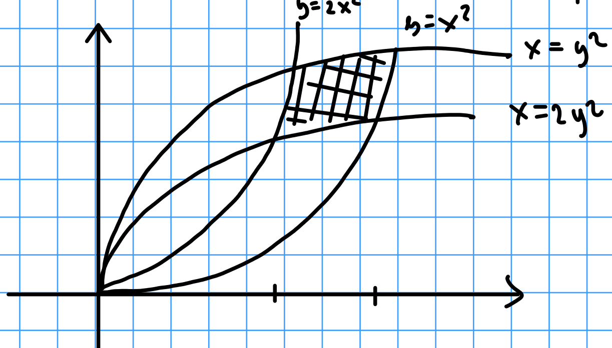
$$\frac{1}{9}\int_0^1 \frac{1-s^2}{\sqrt{s}}\, ds = \frac{1}{9}\left[2\sqrt{s} - \frac{2s^{5/2}}{5}\right]_0^1 = \frac{1}{9}\left(2 - \frac{2}{5}\right) = \frac{8}{45}$$

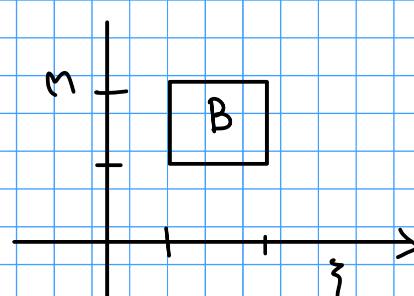
$$\int \left(s^{-1/2} - s^{3/2}\right) ds$$

$$\Rightarrow \iiint_A \frac{xy}{\sqrt{z}}\, dx\,dy\,dz = \boxed{8\sqrt{2} - \frac{8}{45}}$$

<u>ESERCIZIO</u> CALCOLARE L'AREA DI A dove

$$A = \left\{ y^2 \le x \le 2y^2,\quad x^2 \le y \le 2x^2 \right\}$$





CONVIENE INTRODURRE $\xi = \dfrac{x}{y^2}$ $\quad \eta = \dfrac{y}{x^2}$

$A \longrightarrow B = \{ (\xi, \eta) : 1 \leq \xi \leq 2 \ , \ 1 \leq \eta \leq 2 \}$

cerchiamo di usare il teorema di cambio di variabile

Ho definito $\phi(x,y) = \left(\frac{x}{y^2}, \frac{y}{x^2} \right)$ $\quad \phi : A \to B$

$(\phi(A) = B)$

Se applico il teorema di cambio di variabile con questa ϕ con $f(x,y) = 1$

$$\iint_A |\det J_\phi(x,y)| \, dx\, dy = \iint_{\substack{B \\ \| \\ \phi(A)}} d\xi \, d\eta = (2-1)(2-1) = 1$$

NON MI DICE CHI È $|A|$. Devo usare ϕ^{-1} !!

devo esplicitare x e y in termini di (ξ, η)

$$\begin{cases} \xi = \frac{x}{y^2} \\ \eta = \frac{y}{x^2} \end{cases} \qquad \begin{cases} x = \xi y^2 = x = \xi(\eta x^2)^2 = x = \xi \eta^2 x^4 \\ y = \eta x^2 \end{cases}$$

Posso supporre $x \neq 0$ (posso supporlo $(\xi, \eta) \in B$)

$$1 = \xi \eta^2 x^3 \qquad x = \frac{1}{\xi^{\frac{1}{3}} \eta^{\frac{2}{3}}} \implies y = \frac{\eta}{\xi^{\frac{2}{3}} \eta^{\frac{4}{3}}} = \frac{1}{\xi^{\frac{2}{3}} \eta^{\frac{1}{3}}}$$

$$\phi^{-1}(\xi, \eta) = \Bigg(\underbrace{\frac{1}{\xi^{\frac{1}{3}} \eta^{\frac{2}{3}}}}_{\phi_1^{-1}} , \underbrace{\frac{1}{\xi^{\frac{2}{3}} \eta^{\frac{1}{3}}}}_{\phi_2^{-1}} \Bigg)$$

USO LA FORMULA

$$\iint_B |\det J_{\phi^{-1}}(\xi, \eta)| \, d\xi \, d\eta = \iint_{A = \phi^{-1}(B)} dx\, dy = |A|$$

CALCOLO il determinante di $J_{\phi^{-1}}$

$$J_{\phi^{-1}}(\xi,\eta) = \begin{bmatrix} \frac{\partial}{\partial \xi} \frac{1}{\xi^{\frac{1}{3}} \eta^{\frac{2}{3}}} & \frac{\partial}{\partial \eta} \frac{1}{\xi^{\frac{1}{3}} \eta^{\frac{2}{3}}} \\ \frac{\partial}{\partial \xi} \frac{1}{\xi^{\frac{2}{3}} \eta^{\frac{1}{3}}} & \frac{\partial}{\partial \eta} \frac{1}{\xi^{\frac{2}{3}} \eta^{\frac{1}{3}}} \end{bmatrix} =$$

$$\begin{bmatrix} -\frac{1}{3} \frac{1}{\xi^{4/3} \eta^{2/3}} & -\frac{2}{3} \frac{1}{\xi^{1/3} \eta^{5/3}} \\ -\frac{2}{3} \frac{1}{\xi^{5/3} \eta^{1/3}} & -\frac{1}{3} \frac{1}{\xi^{\frac{2}{3}} \eta^{4/3}} \end{bmatrix} \xrightarrow{\det} \frac{1}{9} \frac{1}{\xi^2 \eta^2} - \frac{4}{9} \frac{1}{\xi^2 \eta^2}$$

$$= \boxed{-\frac{1}{3} \frac{1}{\xi^2 \eta^2}}$$

DUNQUE

$$|A| = \iint_B \frac{1}{3} \frac{d\xi \, d\eta}{\xi^2 \eta^2} = \frac{1}{3} \int_1^2 \frac{d\xi}{\xi^2} \int_1^2 \frac{d\eta}{\eta^2} =$$

$$\frac{1}{3} \left[-\xi^{-1} \right]_1^2 \left[-\eta^{-1} \right]_1^2 = \frac{1}{3} \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{2}\right) =$$

$$\frac{1}{3} \frac{1}{2} \frac{1}{2} = \boxed{\frac{1}{12}} = \iint_A 1 = |A|$$

NELLO STESSO MODO POSSO CALCOLARE

$$\iint_A \frac{x^3}{y^3} \, dx \, dy$$

USO IL CAMBIO DI VAR.

$$f(x,y) = \frac{x^3}{y^3}$$

$$\|$$

$$\iint_B f\left(\phi^{-1}(\eta,\xi)\right) \left| \det J_{\phi^{-1}}(\eta,\xi) \right| d\eta \, d\xi =$$

$$\frac{1}{3} \iint_B \frac{\xi}{\eta} \frac{d\xi \, d\eta}{\xi^2 \eta^2} = \frac{1}{3} \int_1^2 \frac{d\xi}{\xi} \int_1^2 \frac{d\eta}{\eta^3} = \frac{1}{3} \left[\ln \xi \right]_1^2 \left[\frac{\eta^{-2}}{-2} \right]_1^2 =$$

$$\frac{1}{3}\ln 2 \ \frac{1}{2}\left(1-\frac{1}{4}\right) = \frac{1}{3}\ln(2)\ \frac{1}{2}\ \frac{3}{4} = \boxed{\frac{\ln(2)}{8}}$$

$$x = \frac{1}{\xi^{\frac{1}{3}}\ \eta^{\frac{2}{3}}} \Rightarrow y = \frac{\eta}{\xi^{\frac{2}{3}}\ \eta^{\frac{4}{3}}} = \frac{1}{\xi^{\frac{2}{3}}\ \eta^{\frac{1}{3}}}$$

$$\left(\frac{x}{y}\right)^3 = \left(\frac{\xi^{\frac{2}{3}}\ \eta^{1/3}}{\xi^{\frac{1}{3}}\ \eta^{2/3}}\right)^3 = \left(\left(\frac{\xi}{\eta}\right)^{1/3}\right) = \frac{\xi}{\eta}$$

$\alpha > 0$

$$f(x,y,z) = \begin{cases} \dfrac{1}{(x^2+y^2+z^2)^{\alpha/2}} & \text{in } B(0,R) = \{x^2+y^2+z^2 \le R^2\} \\ +\infty & \text{se } x=y=z=0 \end{cases}$$

<u>NOTA</u>

$$g(x,y,z) = \frac{1}{x^2+y^2}$$ (la definisco $+\infty$ se il denominatore $=0$) misurabile !!

<u>SI'</u> perché g è continua tranne che in $\{x=y=0\}$ che è una retta in $\mathbb{R}^3$ (l'asse z !). OGNI RETTA È TRASCURABILE IN $\mathbb{R}^3$ (ANCHE IN $\mathbb{R}^2$!!)

- OGNI PIANO È TRASCURABILE IN $\mathbb{R}^3$. DUNQUE la funzione $h(x,y,z) = \dfrac{1}{|x|}$ è misurabile essendo continua tranne che in $\{x=0\}$ = piano(y,z)

I <u>PIANI SONO TRASCURABILI</u> IN $\mathbb{R}^3$ e per vederlo posso usare Tonelli:

$$P = \{(x,y,z) : \quad x = 0\}$$

$$\mathbb{1}_P(x,y,z) = \begin{cases} 0 & \text{se } x \neq 0 \\ 1 & \text{se } x = 0 \end{cases}$$

$$|P| = \iiint_{\mathbb{R}^3} \mathbb{1}_P$$

$$= \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}\left(\underbrace{\int_{-\infty}^{+\infty} \mathbb{1}_P(x,y,z)\,dx}_{=0}\right)dy\right)dz = 0$$

$$\iiint_A \frac{x^2}{x^2+z^2}\,dx\,dy\,dz \qquad \text{dove} \quad A = \{y^2 \geq x^2+z^2, \quad 0 \leq y \leq 1\}$$

OSS $\quad f(x,y,z) := \dfrac{x^2}{x^2+z^2}$ è misurabile (CONTINUA TRANNE CHE SULLA RETTA $x = z = 0$ − ASSE y)

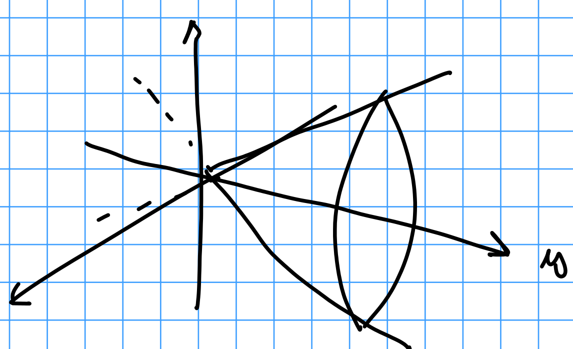
$$(= +\infty \text{ se } x = y = 0)$$

$$f \geq 0$$

INTEGRALE ESISTE (EVENTUALMENTE $+\infty$)

PROVO A DISEGNARE A

$$y^2 = x^2 + z^2 \quad \leftarrow \quad \text{CONO con direttrice asse } Y$$



se $y > 0$

$$y \geq \sqrt{x^2+z^2}$$

CONVIENE PASSARE A COORDINATE "CILINDRICHE" "RISPETTO A Y"

$$\left.\begin{aligned} y &= y \\ x &= \rho\cos\theta \\ z &= \rho\sin\theta \end{aligned}\right\} \phi(y,\rho,\theta) \qquad J_\phi(y,\rho,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\rho\sin\theta \\ 0 & \sin\theta & \rho\cos\theta \end{bmatrix}$$

$\det J_\phi = \rho$ DUNQUE APPLICANDO IL CAMBIO DI VAR.

$$\iiint_A \frac{x^2}{x^2+z^2}\,dx\,dy\,dz = \iiint_{\begin{cases}0\le y\le 1\\ \rho\le y\end{cases}} \left(\frac{\rho^2\cos^2\theta}{\rho^2}\,\rho\right) d\rho\, d\theta\, dy =$$

$$= \int_0^1 \left(\underbrace{\left(\int_0^{2\pi} \cos^2(\theta)\,d\theta\right)}_{=\pi} \int_0^y \rho\, d\rho \right) dy =$$

$$\left[\int_0^{2\pi} \cos^2(\theta)\,d\theta = \int_0^{2\pi} \frac{1+\cos(2\theta)}{2}\,d\theta = \frac{1}{2}\cdot 2\pi = \pi\right]$$

$$= \pi \int_0^1 \left[\frac{\rho^2}{2}\right]_0^y dy = \frac{\pi}{2}\int_0^1 y^2\,dy = \frac{\pi}{2}\left[\frac{y^3}{3}\right]_0^1 = \frac{\pi}{6}$$

#

$$\iint_B \frac{x^2}{x^2+y^2}\left(1-\sqrt{x^2+y^2}\right)dx\,dy \qquad B = B(0,1)$$

COORDINATE POLARI $\quad x=\rho\cos\theta \quad y=\rho\sin\theta$

$dx\,dy = \rho\, d\rho\, d\theta$

$$\iint_{\begin{cases}0\le\theta\le 2\pi\\ 0\le\rho\le 1\end{cases}} \frac{\rho^2\cos^2(\theta)}{\rho^2}(1-\rho)\,\rho\, d\rho\, d\theta =$$

$$\int_0^{2\pi}\cos^2(\theta)\,d\theta \int_0^1 (1-\rho)\rho\, d\rho = \pi\int_0^1(\rho-\rho^2)\,d\rho = \pi\left[\frac{\rho^2}{2}-\frac{\rho^3}{3}\right]_0^1$$

$$= \pi\left(\frac{1}{2}-\frac{1}{3}\right) = \frac{\pi}{6}$$

$$\iint_{B^+} xy\,dx\,dy \qquad B^+ = \{x^2+y^2\le 1 \quad x\ge 0 \quad y\ge 0\}$$

coordinate polari $\longrightarrow$

$$\iint_{\substack{0\le\theta\le\pi/2\\0\le\rho\le 1}} \rho\cos\theta\,\rho\sin\theta\,\rho\,d\theta\,d\rho = \int_0^{\pi/2}\sin\theta\cos\theta\,d\theta\int_0^1\rho^3\,d\rho =$$

$$\frac{1}{2}\int_0^{\pi/2}\sin(2\theta)\,d\theta\left[\frac{\rho^4}{4}\right]_0^1 = \frac{1}{8}\left[\underbrace{\frac{-\cos(2\theta)}{2}}_{=1}\right]_0^{\frac{\pi}{2}} = \frac{1}{8}$$