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Lezioni di Analisi Matematica 2 e Complementi

Lezione 66 13/05/2020

(*) ricevimento il venerdì alle ore 15.00
presso Dip. Matematica (edificio ex Albergo, secondo piano)
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https://people.unipi.it/claudio_saccon/lezioni-di-analisi-2-e-complenti-anno-2019-20/

$$\begin{cases} x' = \boxed{5x - y} + t \\ y' = \boxed{4x + y} - 2t \\ x(0) = 1 \quad y(0) = 0 \end{cases}$$

$$A = \begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}$$

$$\boxed{A = M J M^{-1}}$$

$$M = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \qquad J = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$

$$e^{tA} = M e^{tJ} M^{-1}$$

$$M e^{3t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} = e^{3t} M \begin{bmatrix} -1+2t, & 1-t \\ 2 & -1 \end{bmatrix} =$$

$$e^{3t} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1+2t, & 1-t \\ 2, & -1 \end{bmatrix} = e^{3t} \begin{bmatrix} -1+2t+2, & 1-t-1 \\ -2+4t+2, & 2-2t-1 \end{bmatrix} =$$

$$e^{3t} \begin{bmatrix} 2t+1 & -t \\ 4t & -2t+1 \end{bmatrix}$$

VERIFICA $\qquad \frac{d}{dt} e^{tA} = A e^{tA}$!!

$$\frac{d}{dt} e^{3t} \begin{bmatrix} 2t+1 & -t \\ 4t & -2t+1 \end{bmatrix} = 3e^{3t} \begin{bmatrix} 2t+1 & -t \\ 4t & -2t+1 \end{bmatrix} +$$

$$e^{3t} \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} = e^{3t} \begin{bmatrix} 6t+3+2, & -3t-1 \\ 12t+4, & -6t+3-2 \end{bmatrix} = e^{3t} \begin{bmatrix} 6t+5, & -3t-1 \\ 12t+4, & -6t+1 \end{bmatrix}$$

|| TORNA

$$e^{3t} \begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2t+1 & -t \\ 4t & -2t+1 \end{bmatrix} = e^{3t} \begin{bmatrix} 10t+5-4t, & -5t+2t-1 \\ 8t+4+4t, & -4t-2t+1 \end{bmatrix} = e^{3t} \begin{bmatrix} 6t+5, & -3t-1 \\ 12t+4, & -6t+1 \end{bmatrix}$$

$$e^{tA} = e^{3t} \begin{bmatrix} 2t+1 & -t \\ 4t & -2t+1 \end{bmatrix}$$

Ora $Y(t)$ usando la formula risolutiva $(t_0 = 0)$

$$Y(t) = e^{tA} \left(Y_0 + \int_0^t e^{-\tau A} B(\tau)\, d\tau \right) = \qquad Y_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad B(t) = \begin{bmatrix} t \\ -2t \end{bmatrix}$$

$$e^{tA} Y_0 = e^{3t} \begin{bmatrix} 2t+1 \\ 4t \end{bmatrix}$$

$$e^{tA} \int_0^t e^{-\tau A} B(\tau)\, d\tau = \int_0^t e^{(t-\tau)A} B(\tau)\, d\tau = \qquad \begin{aligned} s &= t-\tau \\ \tau &= t-s \\ d\tau &= -ds \end{aligned}$$

$$-\int_t^0 e^{sA} B(t-s)\,ds = \int_0^t e^{sA} B(t-s)\,ds =$$

$$\int_0^t e^{3s} \begin{bmatrix} 2s+1 & -s \\ 4s & -2s+1 \end{bmatrix} \begin{bmatrix} t-s \\ 2(s-t) \end{bmatrix} ds =$$

$$\int_0^t e^{3s} \begin{bmatrix} (2s+1)(t-s) - 2s(s-t) \\ 4s(t-s) + (1-2s)\,2(s-t) \end{bmatrix} ds = \int_0^t e^{3s} \begin{bmatrix} 2st - 2s^2 + t - s - 2s^2 + 2st \\ 4st - 4s^2 + 2s - 4s^2 - 2t + 4st \end{bmatrix}$$

$$= \int_0^t e^{3s} \begin{bmatrix} -4s^2 + 4st + t - s \\ -8s^2 + 8st + 2s - 2t \end{bmatrix} ds = \cancel{\int_0^t e^{3s}(-4s^2+4st+t-s)\begin{bmatrix} 1 \\ -2 \end{bmatrix} ds}$$

$$\cancel{\left(B(t) = t\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right)}$$

$$\int_0^t e^{3s}(-4s^2+4st+t-s)\,ds =$$

$$\left[\frac{e^{3s}}{3}(-4s^2+4st+t-s)\right]_0^t - \frac{1}{3}\int_0^t e^{3s}(-8s+4t-1)\,ds =$$

$$\frac{1}{3}e^{3t}\left(-4\cancel{t^2}+4\cancel{t^2}+\cancel{t}-\cancel{t}\right) - \frac{t}{3} - \frac{1}{9}\left[e^{3s}(-8s+4t-1)\right]_0^t$$

$$+\frac{1}{9}\int_0^t e^{3s}(-8)\,ds = -\frac{t}{3} - \frac{e^{3t}}{9}(-8t+4t-1) + \frac{1}{9}(4t-1) - \frac{8}{27}\left[e^{3s}\right]_0^t$$

$$-\frac{t}{3} + \frac{e^{3t}}{9}(4t+1) + \frac{4t}{9} - \frac{1}{9} - \frac{8}{27}e^{3t} + \frac{8}{27} =$$

$$\frac{e^{3t}}{27}(12t+3-8) + \frac{t}{9}(-3+4) + \frac{8-3}{27} =$$

$$\frac{e^{3t}}{27}(12t-5) + \frac{t}{9} + \frac{5}{27} = \frac{1}{27}\left[e^{3t}(12t-5) + 3t + 5\right] \leftarrow$$

$$2\int_0^t e^{3s}(-4s^2+4st-t+s)\,ds = 2\left[\frac{e^{3s}}{3}(-4s^2+4st-t+s)\right]_0^t$$

$$-\frac{2}{3}\int e^{3s}(-8s+4t+1)\,ds = \frac{2}{3}e^{3t}\left(-\cancel{4t^2}+\cancel{4t^2}-\cancel{t}+\cancel{t}\right) - \frac{2}{3}(-t)$$

$$-\frac{2}{3}\left[\frac{e^{3s}}{3}(-8s+4t+1)\right]_0^t + \frac{2}{9}\int_0^t e^{3s}(-8)\,ds =$$

$$\frac{2}{3}t - \frac{2}{9}e^{3t}(\underbrace{-8t+4t}_{-4t}+1) + \frac{2}{9}(4t+1) - \frac{16}{27}e^{3t} + \frac{16}{27} =$$

$$\frac{e^{3t}}{27}\left(24t-6-16\right)+\frac{1}{27}\left(18t+24t+6+16\right)=$$

$$\frac{e^{3t}}{27}\left(24t-22\right)+\frac{1}{27}\left(42t+22\right)$$

$$Y(t)=e^{3t}\begin{bmatrix}2t+1\\4t\end{bmatrix}+\frac{e^{3t}}{27}\begin{bmatrix}12t-5\\24t-22\end{bmatrix}+\frac{1}{27}\begin{bmatrix}3t+5\\42t+22\end{bmatrix}=$$

$$\frac{e^{3t}}{27}\begin{bmatrix}54t+27+12t-5\\108t+24t-22\end{bmatrix}+\frac{1}{27}\begin{bmatrix}3t+5\\42t+22\end{bmatrix}=$$

$$\frac{e^{3t}}{27}\begin{bmatrix}66t+22\\132t-22\end{bmatrix}+\frac{1}{27}\begin{bmatrix}3t+5\\42t+22\end{bmatrix}$$

$$\boxed{\begin{aligned}x(t)&=\frac{1}{27}\left(e^{3t}(66t+22)+3t+5\right)\\y(t)&=\frac{1}{27}\left(e^{3t}(132t-22)+42t+22\right)\end{aligned}}$$

VERIFICA

$$x'(t)=\frac{1}{27}\left(3e^{3t}(66t+22)+e^{3t}(66)+3\right)=$$

$$\frac{1}{9}\left(e^{3t}(66t+22+22)+1\right)=\frac{e^{3t}}{9}(66t+44)+\frac{1}{9}$$

$$y'(t)=\frac{1}{27}\left(3e^{3t}(132t-22)+e^{3t}(132)+42\right)=$$

$$\frac{1}{9}\left(e^{3t}(132t-22+44)+14\right)=\boxed{\frac{e^{3t}}{9}(132t+22)+\frac{14}{9}}$$

$y'(t)$

$$5x(t)-y(t)+t=\frac{5}{27}\left(e^{3t}(66t+22)+3t+5\right)$$

$$-\frac{1}{27}\left(e^{3t}(132t-22)+42t+22\right)+t=$$

$$\frac{1}{27}\left\{e^{3t}(330t+110-132t+22)+15t+25-42t-22\right\}+t=$$

$$\frac{1}{27}\left\{e^{3t}(198t+132)-27t+3\right\}=\frac{e^{3t}}{9}(66t+44)-\cancel{t}+\frac{1}{9}=x'(t)$$

$$4x(t) + y(t) - 2t = \frac{4}{27}\left(e^{3t}(66t+22) + 3t+5 \right) + \frac{1}{27}\left(e^{3t}(132t-22) + 42t+22 \right) - 2t =$$

$$\frac{1}{27} e^{3t}\left(264t + 88 + 132t - 22 \right) + \frac{1}{27}\left(12t + 20 + 42t + 22 \right) - 2t =$$

$$\frac{e^{3t}}{27}\left(396t + 66 \right) + \frac{1}{27}\left(54t + 42 \right) - 2t =$$

$$\frac{e^{3t}}{9}\left(132t + 22 \right) + \cancel{2t} + \frac{14}{9} - \cancel{2t} = y'(t)$$

$$\boxed{\text{TORNA}}$$