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Lezioni di Analisi Matematica 2 e Complementi

Lezione 46   23/03/2020

(*) ricevimento il venerdì alle ore 15.00
presso Dip. Matematica (edificio ex Albergo, secondo piano)
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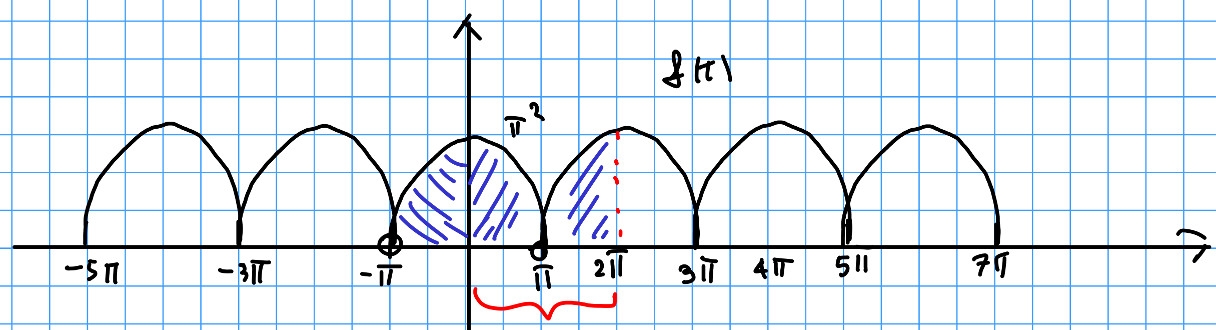
ESEMPI (sviluppi in serie di Fourier)

① $\boxed{f(t) = \pi^2 - t^2}$ se $t \in [-\pi, \pi]$; tale f viene estesa a tutto $\mathbb{R}$ in modo da essere periodica di periodo 2π



(f è regolare a tratti)   f è pari!

Cerco lo sviluppo in serie di Fourier REALE a_n, $b_n = 0$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{2}{2\pi}\int_0^{\pi} f(t)\,dt = \frac{1}{\pi}\int_0^{\pi} (\pi^2 - t^2)\,dt =$$

$$\frac{1}{\pi}\left[\pi^2 t - \frac{t^3}{3}\right]_0^{\pi} = \frac{1}{\pi}\left(\pi^3 - \frac{\pi^3}{3}\right) = \frac{1}{\pi}\,\frac{2\pi^3}{3} = \boxed{\frac{2}{3}\pi^2 = a_0}$$

$n \geq 1$ $$a_n = \frac{\cancel{2}}{\cancel{2}\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt = \frac{2}{\pi}\int_0^{\pi} f(t)\cos(nt)\,dt = \text{(per parti)}$$

$$= \frac{2}{\pi}\Bigg[\underbrace{f(t)\,\frac{\sin(nt)}{n}\Bigg]_0^{\pi}}_{\substack{\sin(n\pi)=0\ \forall n \\ =0}} - \frac{2}{\pi n}\int_0^{\pi} \overbrace{f'(t)}^{-2t}\sin(nt)\,dt = \text{(di nuovo per parti)}$$

$$= -\frac{2}{\pi n}\left[f'(t)\,\frac{(-\cos(nt))}{n}\right]_0^{\pi} - \frac{2}{\pi n^2}\int_0^{\pi} \overbrace{f''(t)}^{-2}\cos(nt)\,dt =$$

$$-\frac{4}{\pi n^2}\Big[t\cos(nt)\Big]_0^{\pi} + \frac{4}{\pi n^2}\int_0^{\pi}\cos(nt)\,dt = -\frac{4}{\pi n^2}\pi\cos(n\pi) + \underbrace{\frac{4}{\pi n^2}\left[\frac{\sin(nt)}{n}\right]_0^{\pi}}_{=0}$$

$$\Rightarrow \boxed{a_n = -\frac{4}{n^2}(-1)^n \qquad \forall n\geq 1}$$

Dato che f è regolare a tratti (ed è continua) $\Rightarrow$ CONV. PUNTUALE A $f(t) = \dfrac{f(t^+)+f(t^-)}{2}$

$$\Rightarrow \boxed{f(t) = \frac{2}{3}\pi^2 - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nt) \qquad \forall t\in\mathbb{R}} \quad (\star)$$

PERÒ posso notare che $|a_n| = \dfrac{4}{n^2}$ dunque $\displaystyle\sum_1^{\infty}|a_n| = \sum_1^{\infty}\frac{4}{n^2} < +\infty$

Allora posso dire che in $(\star)$ ho LA CONVERGENZA UNIFORME

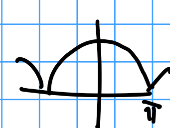
Mettiamo $t=0$ nella $(\star)$.

$$\pi^2 = f(0) = \frac{2}{3}\pi^2 - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\cdot 0) = \frac{2}{3}\pi^2 - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$\Leftrightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = \frac{1}{4}\left(\frac{2}{3}\pi^2 - \pi^2\right) = \frac{1}{4}\left(-\frac{\pi^2}{3}\right) = \boxed{-\frac{\pi^2}{12}}$$

$$\sum_1^{\infty}\frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

Mettiamo $t = \pi$ nella $(\star)$ $\quad f(\pi)=0 \Rightarrow$



$$0 = \frac{2}{3}\pi^2 - 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \underbrace{\cos(n\pi)}_{(-1)^2} = \frac{2}{3}\pi^2 - 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\left((-1)^n(-1)^n = 1 \ \forall n \right)$$

$$\Longleftrightarrow \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{4}\frac{2}{3}\pi^2 = \frac{\pi^2}{6}$$

Per curiosità faccio lo sviluppo complesso:

$$c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-imt}\, dt$$

$m=0 \Rightarrow c_0 = a_0 \ldots$ (come sopra) $= \frac{2}{3}\pi^2$

$m \neq 0$ (posso integrare per parti)

$$c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-imt}\, dt = \underbrace{\left[\frac{1}{2\pi} f(t) \frac{e^{-imt}}{-im}\right]_{-\pi}^{\pi}}_{\text{fa zero perché } f(\pi)=f(-\pi)=0} + \frac{1}{2i\pi m}\int_{-\pi}^{\pi} f'(t)\, e^{-imt}\, dt$$

$$= \frac{1}{2\pi i m}\left[f'(t)\frac{e^{-imt}}{-im}\right]_{-\pi}^{\pi} - \frac{1}{2\pi(im)(-im)}\int_{-\pi}^{\pi} f''(t)\, e^{-imt}\, dt =$$

$$\frac{(-2)}{2\pi}\frac{1}{m^2}\left[t\, e^{-imt}\right]_{-\pi}^{\pi} - \frac{(-2)}{2\pi\, m^2}\int_{-\pi}^{\pi} e^{-imt}\, dt =$$

$$-\frac{1}{\pi}\frac{1}{n^2}\left(\pi e^{-im\pi} - (-\pi) e^{im\pi}\right) + \frac{1}{\pi}\frac{1}{n^2}\underbrace{\left[\frac{e^{-imt}}{-im}\right]_{-\pi}^{\pi}}_{=0} =$$

$$-\frac{1}{m^2}\left(e^{-im\pi} + e^{im\pi}\right) = -\frac{1}{n^2}\, 2\cos(m\pi) = -\frac{2}{m^2}(-1)^n$$

$$\boxed{c_m = \frac{-2}{m^2}(-1)^n}$$

$\left(\text{TORNA: } a_n = 2\,\mathrm{Re}\, c_n = -\frac{4}{n^2}(-1)^n \right)$ ✗

f reale pari $\Rightarrow$ c_m reali pari

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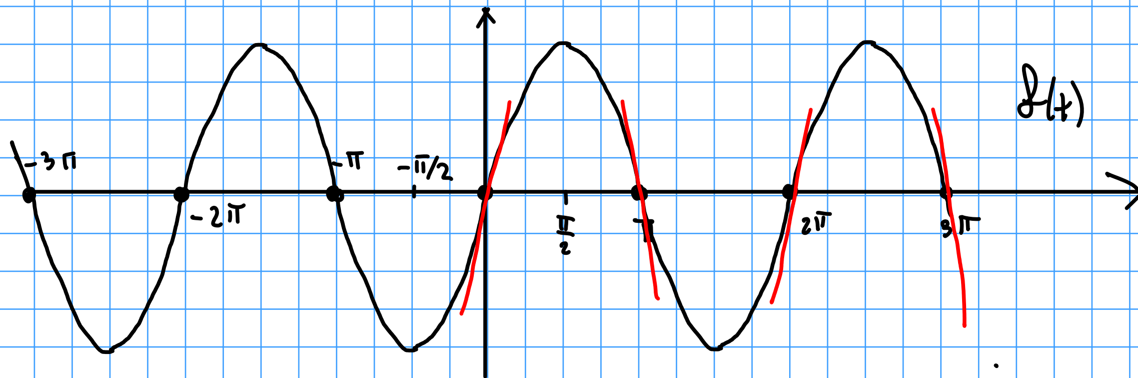
② $f(t) = \begin{cases} t(\pi - t), & \text{se } 0 \le t \le \pi \\ t(\pi + t), & \text{se } -\pi \le t \le 0 \end{cases}$ Inoltre ($f: \mathbb{R} \to \mathbb{R}$) la estendiamo su $\mathbb{R}$ in modo 2π-periodico

Notiamo, se $t \in [0,\pi] \iff -t \in [-\pi, 0]$ e allora

$$f(-t) = (-t)(\pi + (-t)) = -t(\pi - t) = -f(t)$$

f è dispari!!



<u>Nota</u> f è di classe C^1 perché, per esempio in $t=0$ la derivata dx e quella sx coincidono – lo stesso in $t=\pi$ (o $t = k\pi$ $k \in \mathbb{Z}$). In $t=0$:

$$\text{derivata destra} = \frac{d}{dt}\, t(\pi - t)\Big|_{t=0} = \pi - 2t\Big|_{t=0} = \pi$$

$$\text{derivata sinistra} = \frac{d}{dt}\, t(\pi + t)\Big|_{t=0} = \pi + 2t\Big|_{t=0} = \pi$$

Vediamo i coeff. di Fourier di f. <u>REALI</u>

Dato che f è dispari $\Rightarrow a_n = 0$ b_n lo troviamo

(f dispari ⟹ integranda pari)

$$b_n = \frac{\cancel{2}}{\cancel{2}\pi} \int_{-\pi}^{\pi} f(t)\sin(nt)\,dt =$$

$$\frac{2}{\pi}\int_0^{\pi} f(t)\sin(nt)\,dt \underset{\text{(per parti)}}{=} \underbrace{\frac{2}{\pi}\Big[\underset{=0 \text{ se } t=0,\ t=\pi}{f(t)}\frac{\cos(nt)}{-n}\Big]_0^{\pi}}_{=0} + \frac{2}{n\pi}\int_0^{\pi} f'(t)\cos(nt)\,dt$$

$$= \underbrace{\frac{2}{\pi n}\Big[f'(t)\frac{\sin(nt)}{n}\Big]_0^{\pi}}_{=0} - \frac{2}{\pi n^2}\int_0^{\pi} \overset{=-2}{f''(t)}\,\sin(nt)\,dt =$$

$\sin(nt) = 0$ $\forall n \in \mathbb{N}$

$$= \frac{4}{\pi n^2}\Big[\frac{\cos(nt)}{-n}\Big]_0^{\pi} = -\frac{4}{\pi n^3}\big(\cos(n\pi) - \cos(0)\big) = \frac{4(1-(-1)^n)}{\pi n^3}$$

$$b_n = \frac{4(1-(-1)^n)}{n^3\pi} = \begin{cases} 0 & \text{se } n \text{ è pari} \\ \dfrac{8}{(2k+1)^3\pi} & \text{se } n = 2k+1 \text{ (dispari)} \end{cases}$$

Dato che $f \in C^1 \Rightarrow f$ è somma uniforme della sua serie di Fourier. DUNQUE

$$f(t) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(1-(-1)^n)}{n^3}\sin(nt) = \frac{8}{\pi}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^3}\sin((2k+1)t) \qquad (\star)$$

• POSSO ANCHE DIRE che $f'(t) =$ serie delle derivate cioè

$$f'(t) = \frac{8}{\pi}\sum_{k=0}^{\infty} \frac{(2k+1)}{(2k+1)^3}\cos((2k+1)t) = \frac{8}{\pi}\sum_{k=0}^{\infty} \frac{\cos((2k+1)t)}{(2k+1)^3}$$

con convergenza uniforme PERCHÉ

$$\boxed{\sum_{n=0}^{\infty} n|b_n| < +\infty} \Leftarrow \quad n|b_n| = n\left|\frac{4}{\pi}\frac{(1-(-1)^n)}{n^3}\right| \le \frac{4}{\pi}\frac{2}{n^2}$$

(so che $\sum \frac{1}{n^2} < +\infty$)

NOTA
$$f'(t) = \begin{cases} \pi - 2t & 0 \le t \le \pi \\ \pi + 2t & -\pi \le t \le 0 \end{cases} = \pi - 2|t|$$

(è un'onda triangolare: se si confronta questo trovato sopra con i vecchi calcoli, si vede che si è trovato lo stesso risultato)

• Mettiamo $t = \frac{\pi}{2}$ in $(\star)$.

$$\frac{\pi^2}{4} = \frac{\pi}{2}\left(\pi - \frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right) = \frac{8}{\pi}\sum_{k=0}^{\infty} \frac{\sin((2k+1)\pi/2)}{(2k+1)^3} = \frac{8}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} \Leftrightarrow$$

$$\sin\left((2k+1)\frac{\pi}{2}\right) = \sin\left(k\pi + \frac{\pi}{2}\right) = \cos(k\pi)\sin\left(\frac{\pi}{2}\right) + \underbrace{\sin(k\pi)}_{0}\cos\left(\frac{\pi}{2}\right) = (-1)^k$$

$$\boxed{\frac{\pi^3}{32} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3}}$$

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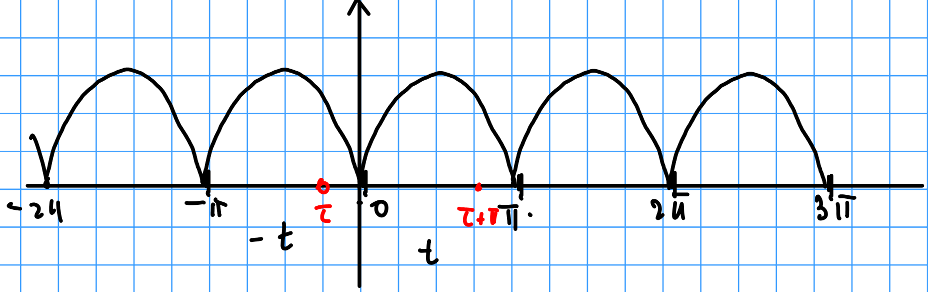
③ $f(t) := \sin(t) \quad 0 \le t \le \pi$. Estendo $f : \mathbb{R} \to \mathbb{R}$ in modo che f sia periodica di periodo π



Continua, non C^1
regolare a tratti

Dato che $T = \pi \Rightarrow \omega = \frac{2\pi}{\pi} = 2$ DUNQUE CERCO LO SVILUPPO DI FOURIER:

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2nt) + b_n \sin(2nt)$$

f È PARI: Verifico che, per $t \in [0,\pi]$, $\boxed{f(-t) = f(t)}$. . INFATTI

se $t \in [0,\pi] \Rightarrow \tau = -t \in [-\pi, 0]$, $\tau + \pi \in [0,\pi)$, $f(-t) = f(\tau+\pi) = \sin(\tau+\pi) = \sin(\pi - t)$

$$= \underbrace{\sin(\pi)}_{0}\cos(-t) + \underbrace{\cos(\pi)}_{-1}\sin(-t) =$$

$$-\sin(-t) = \sin(t) = f(t)$$

$\Rightarrow b_n = 0$. Stavolta parto dalla serie complessa (per vedere questo altro punto di vista)

$$c_0 = \frac{1}{\pi}\int_0^{\pi} f(t)\,dt = \frac{1}{\pi}\int_0^{\pi}\sin(t)\,dt = \frac{1}{\pi}\Big[-\cos(t)\Big]_0^{\pi} = \frac{2}{\pi}$$

se $n \neq 0$
$$c_n = \frac{1}{\pi}\int_0^{\pi} f(t)\, e^{-i2nt}\,dt = \frac{1}{\pi}\int_0^{\pi}\sin(t)\, e^{-2int}\,dt$$

RICORDO $\sin(t) = \frac{e^{it} - e^{-it}}{2i} \Rightarrow$

$$c_n = \frac{1}{2\pi i}\left(\int_0^{\pi} e^{it} e^{-2int}\,dt - \int_0^{\pi} e^{-it} e^{-2int}\,dt\right) =$$

$$\frac{1}{2\pi i}\left(\int_0^{\pi} e^{(1-2n)it}\,dt - \int_0^{\pi} e^{-(1+2n)it}\,dt\right) =$$

$$\frac{1}{2\pi i}\left[\frac{e^{(1-2n)it}}{(1-2n)i} - \frac{e^{-(1+2n)it}}{-(1+2n)i}\right]_0^{\pi} =$$

$$-\frac{1}{2\pi}\left[\frac{e^{(1-2n)it}}{1-2n}+\frac{e^{-(1+2n)it}}{1+2n}\right]_0^{\pi} =$$

$$-\frac{1}{2\pi}\left[e^{-2nit}\left(\frac{e^{it}}{1-2n}+\frac{e^{-it}}{1+2n}\right)\right]_0^{\pi} =$$

$$-\frac{1}{2\pi}\left\{\underbrace{e^{-2n\pi i}}_{=\,e^0}\left(\frac{\overset{-1}{e^{i\pi}}}{1-2n}+\frac{\overset{-1}{e^{-i\pi}}}{1+2n}\right) - e^{0}\left(\frac{1}{1-2n}+\frac{1}{1+2n}\right)\right\} =$$

e^{ix} è periodico di periodo 2π

$$-\frac{1}{2\pi}\left\{-1\left(\frac{1}{1-2n}+\frac{1}{1+2n}\right) - 1\left(\frac{1}{1-2n}+\frac{1}{1+2n}\right)\right\} =$$

$$\frac{1}{\pi}\left(\frac{1}{1-2n}+\frac{1}{1+2n}\right) = \frac{1}{\pi}\,\frac{2}{1-4n^2}$$

$$\boxed{c_n = \frac{2}{\pi}\,\frac{1}{1-4n^2} \qquad \forall n\in\mathbb{Z}}$$

i c_n sono reali pari!! (TORNA) $\Rightarrow$ $b_n=0$

$a_0 = c_0$ $\qquad$ $a_n = 2c_n = \dfrac{4}{\pi}\,\dfrac{1}{1-4n^2}$ $\quad n\neq 0$, $n\in\mathbb{N}$

Vediamo come ci si arriva facendo integrali reali.

$b_n = 0$ / $a_0 = c_0$ visto /

$$n \geq 1 \qquad a_n = \frac{2}{\pi}\int_0^{\pi} f(t)\cos(2nt)\,dt = \frac{2}{\pi}\int_0^{\pi}\sin(t)\cos(2nt)\,dt$$

$$\frac{\pi}{2}a_n = \int_0^{\pi}\sin(t)\cos(2nt)\,dt \overset{\text{(per parti)}}{=} \underbrace{\left[\sin(t)\,\frac{\sin(2nt)}{2n}\right]_0^{\pi}}_{=0} +$$

$$-\frac{1}{2n}\int_0^{\pi}\cos(t)\sin(2nt)\,dt \overset{\text{(per parti)}}{=} \underbrace{-\frac{1}{2n}\left[\cos(t)\left(\frac{-\cos(2nt)}{2n}\right)\right]_0^{\pi}}_{} +$$

$$-\frac{1}{4n^2}\int_0^{\pi}-\sin(t)\cos(2nt)\,dt = \frac{1}{4n^2}\left[\cos(t)\cos(2nt)\right]_0^{\pi} + \frac{1}{4n^2}\,\frac{\pi}{2}a_n$$

$$= \frac{1}{4n^2}\left(-\underset{=1}{\cos(2n\pi)} - 1\right) + \frac{1}{4n^2}\,\frac{\pi}{2}a_n = -\frac{1}{2n^2} + \frac{1}{4n^2}\,\frac{\pi}{2}a_n \qquad \text{DUNQUE}$$

$$\left(\frac{1}{4n^2}-1\right)\frac{\pi}{2}a_n = \frac{1}{2n^2} \iff \frac{\pi}{2}a_n = \frac{\frac{1}{2n^2}}{\left(\frac{1}{4n^2}-1\right)} = \frac{\frac{1}{2n^2}}{\frac{1-4n^2}{4n^2}} =$$

$$= \frac{2}{1-4n^2} \iff \boxed{a_n = \frac{4}{\pi}\,\frac{1}{1-4n^2}}$$

Dato che f è regolare a tratti ed è continua;

$$f(t) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{1-4n^2}\cos(2nt) \qquad \forall t\in\mathbb{R}$$

IN PART.
$$\boxed{\sin(t) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2nt)}{4n^2-1} \qquad \forall t\in[0,\pi]}$$

(e la conv. è uniforme dato che $|a_n| \simeq \frac{1}{4n^2}$ ...).

- Mettiamo $t=\frac{\pi}{2}$ nella formula sopra:

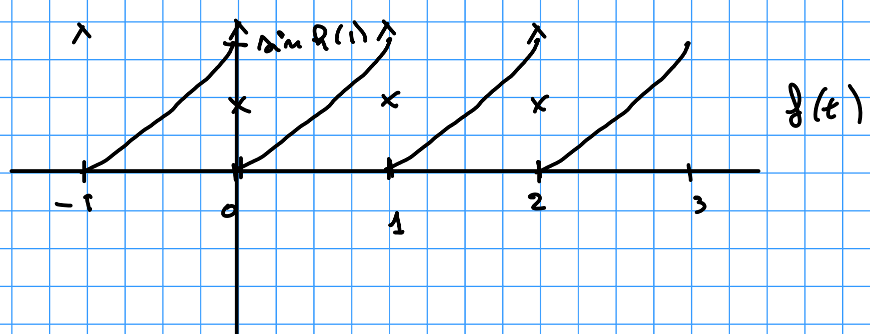
$$1 = \sin\left(\frac{\pi}{2}\right) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{1}^{\infty}\frac{(-1)^n}{4n^2-1} \quad \leftarrow \cos(n\pi) \qquad \text{DUNQUE}$$

$$\sum_{1}^{\infty}\frac{(-1)^n}{4n^2-1} = \left(\frac{2}{\pi}-1\right)\frac{\pi}{4} = \frac{1}{2}-\frac{\pi}{4} = \frac{2-\pi}{4} \quad (<0)$$

- Mettiamo $t=0 \implies$

$$0 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{1}^{\infty}\frac{1}{4n^2-1} \iff \sum_{1}^{\infty}\frac{1}{4n^2-1} = \frac{1}{2} \qquad \#$$

(4) $f(t) = \sinh(t) \quad 0\le t\le 1$, $\quad f:\mathbb{R}\to\mathbb{R}$ 1-periodica



$T=1 \qquad \omega = 2\pi$

Né pari né dispari

Uso lo sviluppo complesso.

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2}$$

$$c_0 = \frac{1}{1}\int_0^1 f(t)\,dt = \int_0^1 \sinh(t)\,dt = \Big[\cosh(t)\Big]_0^1 = \cosh(1) - 1$$

$$= \boxed{\frac{e^{1} + e^{-1} - 2}{2} = c_0}$$

$$n \neq 0 \quad c_n = \frac{1}{1}\int_0^1 f(t)\,e^{-2\pi n i t}\,dt = \int_0^1 \frac{e^{(1-2\pi i n)t} - e^{-(1+2\pi i n)t}}{2}\,dt$$

$$= \frac{1}{2}\left[\frac{e^{(1-2\pi i n)t}}{1-2\pi i n} - \frac{e^{-(1+2\pi i n)t}}{-(1+2\pi i n)}\right]_0^1 =$$

$$\frac{1}{2}\left(\frac{e^{1-2\pi i n}}{1-2\pi i n} + \frac{e^{-1-2\pi i n}}{1+2\pi i n} - \frac{1}{1-2\pi i n} - \frac{1}{1+2\pi i n}\right) =$$

$$\frac{1}{2}\left(\frac{e}{1-2\pi i n} + \frac{e^{-1}}{1+2\pi i n} - \frac{2}{1+4\pi^2 n^2}\right) =$$

$$e^{1-2\pi i n} = e^{1}\underbrace{e^{-2\pi i n}}_{=1} = e^{1}$$

$$\frac{1}{2}\left(\frac{e + 2\pi e i n + e^{-1} - 2\pi e^{-1} i n - 2}{1+4\pi^2 n^2}\right) =$$

$$\frac{1}{1+4\pi^2 n^2}\left(\frac{e + e^{-1} - 2}{2} + i\pi n\left(e - e^{-1}\right)\right) =$$

$$\boxed{\frac{\cosh(1) - 1}{1+4\pi^2 n^2} + \frac{i\,2\pi \sinh(1)}{1+4\pi^2 n^2} = c_n}$$

(noto che se $n=0$ trovo $c_0 = a_0$)

Ne ricavo:

$$\boxed{a_n = \frac{2(\cosh(1) - 1)}{1+4\pi^2 n^2} \qquad b_n = \frac{-4\pi \sinh(1)\,n}{1+4\pi^2 n^2} \qquad n \neq 0}$$

<u>NOTA</u> $\sum |a_n| < +\infty \quad \left(|a_n| \simeq \frac{c}{n^2}\right) \qquad \sum |b_n| = +\infty \qquad |b_n| \simeq \frac{1}{n}$

TORNA CON IL FATTO CHE f è discontinua in $1, 2, 3 \ldots$

Per i teorema posso dire che

$$\frac{f(t^+) + f(t^-)}{2} = a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi n t) + b_n \sin(2\pi n t)$$

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