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Lezioni di Analisi Matematica 2 e Complementi

Lezione 68 1/6/18

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$$\begin{aligned} x' &= -5x + y \\ y' &= -x - 3y \\ z' &= -x + y - 4z \end{aligned}$$

$x(0) = 1$

$y(0) = -1$

$z(0) = 0$

PROBLEMA OMOGENEO

$$A = \begin{bmatrix} -5 & 1 & 0 \\ -1 & -3 & 0 \\ -1 & 1 & -4 \end{bmatrix}$$

$$p(\lambda) = (-4-\lambda)\det\begin{bmatrix} -5-\lambda & 1 \\ -1 & -3-\lambda \end{bmatrix} =$$

$$-(4+\lambda)\left((5+\lambda)(3+\lambda)+1\right) =$$

$$-(4+\lambda)(\lambda^2+8\lambda+16) = -(4+\lambda)^3$$

$\lambda = -4 \qquad m_A = 3$

$$B = A + 4I = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$

RANGO 1 $\quad \ker \to \dim 2$

$m_G = 2 \quad \Rightarrow \quad B^2 = 0$

e_3

$B \downarrow$

$e_2 \quad e_1$

$\downarrow \quad \downarrow$

$0 \quad 0$

DEVO TROVARE $e_3 \neq 0$ con $Be_3 \neq 0$

$e_2 := Be_3 \in \ker B$, $e_1 \in \ker B$ indipendente da e_2

Prendo $e_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow e_2 = Be_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

oppure $e_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow e_2 = Be_3 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$

$e_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow e_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Mi serve $e_1 \in \ker B$ indip da e_2 $\quad e_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ deve essere

$x = y$ cioè $e_1 = \begin{pmatrix} \alpha \\ \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ se $\beta = 0$ e_1 è indip. da e_2

andrebbe bene anche $e_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$e_3 \downarrow$

$e_1 \quad e_2$

Con le scelte fatte

$$M = [e_1 | e_2, e_3] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad e \quad J = \left[\begin{array}{c|cc} -4 & 0 & 0 \\ \hline 0 & -4 & 1 \\ 0 & 0 & -4 \end{array}\right]$$

se prendo $M = [e_2, e_3, e_1] \Rightarrow J = \begin{bmatrix} -4 & 1 & 0 \\ 0 & -4 & 0 \\ 0 & & -4 \end{bmatrix}$

($M = [e_3, e_2, e_1] \Rightarrow$ "J" $= \begin{bmatrix} -4 & 0 & 0 \\ 1 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix}$ No)

Calcoliamo M^{-1} $\quad \det M = -1 = (-1)\det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1} = -1 \; \mathrm{cof}(M^t) = -\mathrm{cof}(M) = -\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

$(M^t = M)$

$$M^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \quad \Rightarrow \quad e^{tJ} = e^{-4t} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

$$e^{tA} = e^{-4t} M \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} =$$

$$e^{-4t} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ -t & t & 1 \\ -1 & 1 & 0 \end{bmatrix} =$$

$$e^{tA} = e^{-4t} \overbrace{\begin{bmatrix} 1-t & t & 0 \\ -t & 1+t & 0 \\ -t & t & 1 \end{bmatrix}}^{M(t)} = e^{-4t} M(t)$$

VERIFICHIAMO CHE $\frac{d}{dt} e^{tA} = A e^{tA}$

$$\frac{d}{dt} e^{-4t} M(t) = -4 e^{-4t} M(t) + e^{-4t} M'(t) =$$

$$e^{-4t} \left\{ \begin{bmatrix} -4+4t & -4t & 0 \\ 4t & -4-4t & 0 \\ 4t & -4t & -4 \end{bmatrix} + \begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \right\} =$$

$$e^{-4t} \begin{bmatrix} -5+4t & 1-4t & 0 \\ -1+4t & -3-4t & 0 \\ -1+4t & 1-4t & -4 \end{bmatrix}$$

$$e^{-4t} A M(t) = e^{-4t} \begin{bmatrix} -5 & 1 & 0 \\ -1 & -3 & 0 \\ -1 & 1 & -4 \end{bmatrix} \begin{bmatrix} 1-t & t & 0 \\ -t & 1+t & 0 \\ -t & t & 1 \end{bmatrix} =$$

$$e^{-4t}\begin{bmatrix} -5+5t-t & -5t+1+t & 0 \\ -1+t+3t & -t-3-3t & 0 \\ -1+t-t+4t & -t+1+t-4t & -4 \end{bmatrix} = e^{-4t}\begin{bmatrix} -5+4t & 1-4t & 0 \\ -1+4t & -3-4t & 0 \\ -1+4t & 1-4t & -4 \end{bmatrix}$$

OK

$$\begin{bmatrix} -5+4t & 1-4t & 0 \\ -1+4t & -3-4t & 0 \\ -1+4t & 1-4t & -4 \end{bmatrix}$$

TORNIAMO AL PROBLEMA : $Y_0 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$$Y(t) = e^{-4t}\begin{bmatrix} 1-t & t & 0 \\ -t & 1+t & 0 \\ -t & t & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \left(e^{tA}Y_0\right)$$

$$e^{-4t}\begin{bmatrix} 1-2t \\ -1-2t \\ 2t \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

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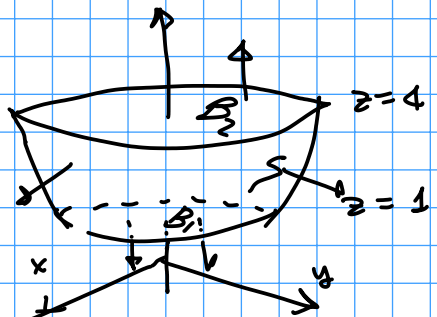
$$\vec{f} = 10x^3y^2z\,\vec{\imath} - 2x^2y^3z^2\,\vec{\jmath} + x^2y^2(2z^3 - 15z^2 + 24z)\,\vec{k}$$

$$D = \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \le z,\ 1 \le z \le 4\right\}$$

(1) Voglio $\iint_{\partial D} \vec{f}\cdot\hat{\nu}\,d\sigma$ con $\hat{\nu}$ normale uscente da D

$$\partial D = B_1 \cup S \cup B_2$$

$B_1 = \{x^2+y^2 \le 1,\ z=1\}$ $\quad \hat{\nu} = -\vec{k}$



$B_2 = \{ x^2+y^2 \le 4,\ z=4 \}$ $\quad \hat{\nu} = \vec{k}$

$S = \{ x^2+y^2 = z \quad 1 \le z \le 4 \}$ $\quad \hat{\nu}$ punta "verso l'esterno"

VIENE NATURALE CERCARE LA DIVERGENZA.

$$\operatorname{div} \vec{f} = 30x^2y^2z - 6x^2y^2z^2 + x^2y^2(6z^2 - 30z + 24) = 24x^2y^2$$

APPLICO IL TEOR. DIV. $\Rightarrow$

$$\text{FLUSSO} = 24 \iiint_D x^2 y^2 \, dx\,dy\,dz =$$

$$24 \int_1^4 dz \iint_{\{x^2+y^2 \le z\}} x^2 y^2 \, dx\,dy = \quad (\text{coord. polari in } x, y)$$

$$24 \int_1^4 dz \int_0^{2\pi} d\theta \int_0^{\sqrt{z}} \rho^2 \cos^2\theta \, \rho^2 \sin^2\theta \, \rho \, d\rho =$$

$$24 \int_1^4 \left(\int_0^{2\pi} \cos^2\theta \sin^2\theta \, d\theta \int_0^{\sqrt{z}} \rho^5 \, d\rho \right) dz = \otimes$$

HA INTEGRALE NULLO

$$\int_0^{2\pi} \cos^2\theta \sin^2\theta = \frac{1}{4} \int_0^{2\pi} \sin^2(2\theta) \, d\theta = \frac{1}{4} \int_0^{2\pi} \frac{1-\cos(4\theta)}{2} \, d\theta = \frac{2\pi}{8} = \frac{\pi}{4}$$

$$\otimes = 6\pi \int_1^4 \left[\frac{\rho^6}{6} \right]_0^{\sqrt{z}} dz = \pi \int_1^4 z^3 \, dz = \pi \left[\frac{z^4}{4} \right]_1^4 =$$

$$\left(\frac{255}{4} \right) \pi$$

(2) VOGLIO ANCHE $\iint_S \vec{f} \cdot \hat{\nu} \, d\sigma$ (con lo stesso $\hat{\nu}$ di prima)

SO CHE: $\iint_S \vec{f} \cdot \hat{\nu} \, d\sigma + \iint_{B_1} \vec{f} \cdot \hat{\nu} \, d\sigma + \iint_{B_2} \vec{f} \, \hat{\nu} \, d\sigma = \frac{255}{4} \pi$

$$\iint_{B_1} \vec{f} \cdot \hat{\nu} \, d\sigma = \iint_{\{x^2+y^2 \le 1\}} f_3(x, y, 1)(-1) \, dx\,dy =$$

$$-\iint_{\{x^2+y^2\le 1\}} x^2y^2\,(2-15+24)\,dx\,dy = -11\iint_{\{x^2+y^2\le 1\}} x^2y^2\,dx\,dy =$$

$$-11\int_0^{2\pi}\cos^2\theta\sin^2\theta\,d\theta\int_0^1\rho^5\,d\rho = -11\,\frac{\pi}{4}\,\frac{1}{6} = -\frac{11}{24}\pi$$

$$\iint_{B_2}\vec{f}\cdot\hat{\nu}\,d\sigma = \iint_{\{x^2+y^2\le 4\}} f_3(x,y,4)\,dx\,dy =$$

$$-16\iint_{\{x^2+y^2\le 4\}} x^2y^2\,dx\,dy = -16\int_0^{2\pi}\cos^2\theta\sin^2\theta\int_0^2\rho^5\,d\rho = -16\,\frac{\pi}{4}\,\frac{2^6}{6} = -\frac{128}{3}\pi$$

$$\left(2\cdot 4^3 - 15\,4^2 + 24\cdot 4 = 2\cdot 4^3 - 15\cdot 4^2 + 6\cdot 4^2 = 4^2\left(8-15+6\right) =\right)$$

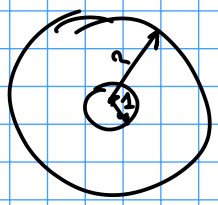
Se i conti sono giusti viene $\iint_S \vec{f}\cdot\hat{\nu}\,d\sigma = \left(\frac{255}{4}+\frac{11}{24}+\frac{128}{3}\right)\pi = \dots$

CALCOLIAMO L'AREA DI S

$$S = \{\, z = x^2+y^2 \qquad 1\le z\le 4\,\}$$

S è grafico di $g(x,y) = x^2+y^2$ su $\Omega = \{\, 1 \le x^2+y^2 \le 4\,\}$

$\Omega = B(0,2)\setminus B(0,1)$



$$A(S) = \int_\Omega \sqrt{1+|\nabla g|^2}\,dx\,dy = \iint_\Omega \sqrt{1+4x^2+4y^2}\,dx\,dy =$$

$$\int_0^{2\pi} d\theta\int_1^2 \sqrt{1+4\rho^2}\;\rho\,d\rho = 2\pi\int_1^4 \frac{1}{2}\sqrt{1+4s}\,ds \qquad (s=\rho^2) =$$

$$\pi\left[\frac{2}{3}(1+4s)^{\frac{3}{2}}\,\frac{1}{4}\right]_1^4 = \frac{1}{6}\pi\left[(1+4s)^{3/2}\right]_1 = \frac{\pi}{6}\left(17^{3/2}-5^{3/2}\right)$$

$$\begin{cases} x' = 2x - y \\ y' = x \\ z' = 2x - 2y - z \end{cases} \qquad A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ 2 & -2 & -1 \end{bmatrix}$$

$$p(\lambda) = (2-\lambda)\det\begin{bmatrix} -\lambda & 0 \\ -2 & -1-\lambda \end{bmatrix} + \det\begin{bmatrix} 1 & 0 \\ 2 & -1-\lambda \end{bmatrix} =$$

$$(2-\lambda)\lambda(1+\lambda) - (1+\lambda) = (1+\lambda)(2\lambda - \lambda^2 - 1) =$$

$$-(1+\lambda)(1-\lambda)^2$$

$\lambda_1 = -1 \qquad \lambda_2 = 1$ (doppio)

$\lambda = \lambda_2 = 1$ $\qquad B = A - I = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 2 & -2 & -2 \end{bmatrix}$ rango 2

$\lambda_1 \quad \lambda_2$

e_3
$\downarrow B$
$e_1 \quad e_2$

Mi serve $e_3 \in \ker B^2 \setminus \ker B$

$$B^2 = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 2 & -2 & -2 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 2 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -4 & 4 & 4 \end{bmatrix}$$

$\ker B^2 \rightarrow \quad -x + y + z = 0 \qquad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = e_3$

$$Be_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = e_2 \neq 0$$

Mi serve e_1 autovettore relativo a $\lambda_1 = -1$ cioè nel $\ker$ di

$$A + I = \begin{bmatrix} 3 & -1 & 0 \\ 1 & 1 & 0 \\ 2 & -2 & 0 \end{bmatrix} \qquad 0 = x = y \quad z \text{ qualsiasi} \qquad e_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$M = \begin{bmatrix} e_1, e_2, e_3 \end{bmatrix} = \begin{bmatrix} 6 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad J = \begin{pmatrix} -1 & 0 & 0 \\ 6 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det M = -1 \qquad M^{-1} = -\operatorname{adj}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = -\begin{pmatrix} 1 & -1 & -1 \\ 0 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix}$$

$$M^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

e^{tJ}

$$e^{tA} = M \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{t} & te^{t} \\ 0 & 0 & e^{t} \end{bmatrix} M^{-1} =$$

$$M \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{t} & te^{t} \\ 0 & 0 & e^{t} \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -e^{-t} & e^{-t} & e^{-t} \\ te^{t} & e^{t}(1-t) & 0 \\ e^{t} & -e^{t} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} e^{t}(1+t) & -te^{t} & 0 \\ te^{t} & e^{t}(1-t) & 0 \\ e^{t}-e^{-t} & e^{-t}-e^{t} & e^{-t} \end{bmatrix}$$

PROVO CON $Y_0 = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$

$$Y(t) = e^{tA} Y_0 = \begin{bmatrix} 2e^{t}(1+t) \\ 2te^{t} \\ 2e^{t} \end{bmatrix}$$

PROVIAMO A FARE LA VERIFICA

$x(t) = 2e^{t}(1+t)$ $\quad x'(t) = 2e^{t}(1+t) + 2e^{t} = 2e^{t}(2+t)$

$y(t) = 2te^{t}$ $\quad y'(t) = 2te^{t} + 2e^{t} = 2e^{t}(1+t)$

$z(t) = 2e^{t}$ $\quad z'(t) = 2e^{t}$

$$\left(\begin{cases} x' = 2x - y \\ y' = x \\ z' = 2x - 2y - z \end{cases} \right)$$

$2x - y = 2e^t(2 + 2t - t) = 2e^t(2 + t) = x'(t)$

$x = 2e^t(1 + t) = y'(t)$

$2x - 2y - z = 2e^t(2 + 2t - 2t - 1) = 2e^t = z'(t)$

TORNA