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Lezioni di Analisi Matematica 2 e Complementi

Lezione 67 30/5/18

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$$\begin{cases} x' = -3x + y \\ y' = -x - y + e^{-2t} \end{cases} \qquad \begin{matrix} x(0)=0 \\ y(0)=0 \end{matrix}$$

(NON OMOGENEO)

$$A = \begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix} \qquad B(t) = \begin{pmatrix} 0 \\ e^{-2t} \end{pmatrix}$$

$$p(\lambda) = (-3-\lambda)(-1-\lambda) + 1 = 3 + 4\lambda + \lambda^2 + 1 = \lambda^2 + 4\lambda + 4 = (\lambda+2)^2$$

$\lambda = -2$ DOPPIO $(m_A = 2)$

$$B = A + 2I = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \neq 0 \text{, rango 1, } \ker \dim \to 1$$

$m_G = 1 < m_a$

$e_2 \xrightarrow{B} e_1 \xrightarrow{B} 0$ Devo trovare e_2 tale che $Be_2 \neq 0$ ($Be_2 = e_1$, e_1 autovalore)

$$e_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad Be_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \qquad (Be_2 = 0)$$

$$M = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \qquad M^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \qquad J = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}$$

$$e^{tA} = e^{-2t}\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = e^{-2t}\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} t & -1-t \\ 1 & -1 \end{pmatrix} =$$

$$e^{-2t}\begin{pmatrix} -t+1 & 1+t-1 \\ -t & 1+t \end{pmatrix} = e^{-2t}\begin{pmatrix} 1-t & t \\ -t & 1+t \end{pmatrix}$$

$$Y(t) = e^{tA}\left(Y_0 + \int_0^t e^{-sA}B(s)\,ds\right) = e^{tA}\int_0^t e^{2s}\begin{pmatrix} 1+s & -s \\ +s & 1-s \end{pmatrix}\begin{pmatrix} 0 \\ e^{-2s} \end{pmatrix} ds$$

($Y_0 = 0$)

$$e^{tA}\int_0^t e^{2s}\begin{pmatrix} -s\,e^{-2s} \\ (1-s)\,e^{-2s} \end{pmatrix} ds = e^{tA}\int_0^t \begin{pmatrix} -s \\ 1-s \end{pmatrix} ds =$$

$$e^{tA}\begin{pmatrix} -t^2/2 \\ t - t^2/2 \end{pmatrix} = e^{-2t}\begin{pmatrix} 1-t & t \\ -t & 1+t \end{pmatrix}\begin{pmatrix} -t^2/2 \\ t-t^2/2 \end{pmatrix} =$$

$$e^{-2t}\begin{pmatrix} -(1-t)\frac{t^2}{2} + t\left(t - t^2/2\right) \\ t^3/2 + (1+t)\left(t - t^2/2\right) \end{pmatrix} = e^{-2t}\begin{pmatrix} -t^2/2 + t^3/2 + t^2 - t^3/3 \\ t^3/2 + t - t^2/2 + t^2 - t^3/2 \end{pmatrix} =$$

$$e^{-2t}\begin{pmatrix} t^2/2 \\ t + t^2/2 \end{pmatrix} \Rightarrow \quad x(t) = \frac{t^2}{2}e^{-2t}, \quad y(t) = \left(t + \frac{t^2}{2}\right)e^{-2t}$$

VERIFICA (NON NECESSARIA, MA...)

$$x'(t) = t\,e^{-2t} + (-2)\frac{t^2}{2}e^{-2t} = \left(t - t^2\right)e^{-2t}$$

$$-3x(t) + y(t) = \left(-\frac{3t^2}{2} + t + \frac{t^2}{2}\right)e^{-2t} = \left(t - t^2\right)e^{-2t}$$

ok

$$y'(t) = (1+t)e^{-2t} - 2\left(t + t^2/2\right)e^{-2t} = \left(1 - t - t^2\right)e^{-2t}$$

$$-x(t) - y(t) + e^{-2t} = \left(-\frac{t^2}{2} - t - \frac{t^2}{2} + 1\right)e^{-2t} = \left(1 - t - t^2\right)e^{-2t}$$

ok

$x(0) = 0 \qquad y(0) = 0$

$$\begin{cases} x' = 3x - y \\ y' = x + y \\ z' = y + z \end{cases} \qquad A = \begin{pmatrix} 3 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$p(\lambda) = \det\begin{pmatrix} 3-\lambda & -1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 1 & 1-\lambda \end{pmatrix} = (1-\lambda)\det\begin{pmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{pmatrix} = (1-\lambda)\left((3-\lambda)(1-\lambda)+1\right) =$$

$$(1-\lambda)\left(\lambda^2 - 4\lambda + 3 + 1\right) = (1-\lambda)(\lambda-2)^2$$

$\lambda = 1$ semplice $\qquad \lambda = 2$ doppio

$$B = A - 2I = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

rango 2 $\quad m_G(2) = 1$

$\dim \operatorname{Ker}(B) = 1$

$\lambda = 2 \qquad \lambda = 1$

e_3

$\downarrow$

$e_2 \qquad e_1$

$$J = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Devo trovare e_3 tale che $B^2 e_3 = 0 \quad B e_3 \neq 0 \quad (e_2 = B e_3)$

(e poi e_1 nel $\operatorname{Ker}(A - I)$) $\qquad$ Mi serve B^2 !!

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -2 & 1 \end{pmatrix}$$

$\operatorname{Ker} B^2 = \{ x - 2y + z = 0 \}$

MI SERVE $\quad y = \dfrac{x+z}{2} \quad$ e $\quad \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \neq 0$

$$e_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad B e_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} =: e_2$$

Cerco e_1 nel Ker di $\begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad e_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$M = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \qquad J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$\left(M \hat{e}_3 = e_3 \quad \text{per come è fatto } M \right.$

$\Downarrow$

$\left. M^{-1} e_3 = \hat{e}_3 \right)$

Cerco la soluzione con $Y(0) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

NOTO CHE $\quad Y(0) = -e_3 \quad \Rightarrow \quad M^{-1} Y_0 = -\hat{e}_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

$$\Rightarrow Y(t) = M e^{tJ} \hat{e}_3 = M \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} =$$

$$-e^{2t} M \begin{pmatrix} 0 \\ t \\ 1 \end{pmatrix} = -e^{2t}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ t \\ 1 \end{pmatrix} = -e^{2t}\begin{pmatrix} t+1 \\ t \\ t-1 \end{pmatrix}$$

ES. $\qquad Y' = AY \quad$ dove

$$A = \begin{pmatrix} 3 & -1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} \rightarrow p(\lambda) = -(\lambda-2)^3$$

$(2-\lambda)\det\begin{vmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} =$

$\lambda = 3$ Tripla

$$B = A - 2I = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$ rango 2 (ker dim 1)

$$\begin{pmatrix} e_3 \\ \downarrow B \\ e_2 \\ \downarrow B \\ e_1 \end{pmatrix} \quad B^2 \quad J = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

($B^3 e_3 = 0$ ovvio)

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & -2 & 0 \end{pmatrix}$$

Mi serve e_3 t.c. $B^2 e_3 \neq 0$

$$e_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad B^2 e_3 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} =: e_1 \qquad B e_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} =: e_2$$

$$M = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix} \qquad M^{-1} = -\frac{1}{2} \text{ matr. dei complementi di } \begin{pmatrix} 0 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} =$$

$$-\frac{1}{2}\begin{pmatrix} 0 & +1 & -1 \\ 0 & -2 & 0 \\ -2 & 2 & 0 \end{pmatrix}$$

$$e^{tA} = \frac{e^{2t}}{2}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & 1 \\ 0 & 2 & 0 \\ 2 & -2 & 0 \end{pmatrix} \quad - \quad -$$

FINIRE I CALCOLI e TROVARE LA SOL CON $Y(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

LEZIONE DI RECUPERO VENERDÌ ORE 9.30 aula C11