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Lezioni di Analisi Matematica 2 e Complementi

Lezione 66 29/5/18

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CASO NON OMOGENEO:

$$\begin{cases} Y' = AY + B(x) \\ Y(0) = Y_0 \end{cases}$$

$B: I \to \mathbb{R}^N$ continua, A matrice $N \times N$.

$0 \in I$ (se no si può traslare)

$$\Rightarrow \quad Y(t) = e^{tA}\left(Y_0 + \int_0^t e^{-sA} B(s)\, ds\right)$$

Verifichiamo che la formula dà una soluzione:

(1) $Y(0) = e^{0A}\left(Y_0 + \int_0^0 e^{-sA} B(s)\, ds\right) = I \cdot Y_0 = Y_0$

(2) $Y'(t) = A e^{tA}\left(Y_0 + \int_0^t e^{-sA} B(s)\, ds\right) + e^{tA}\left(Y_0 + \int_0^t e^{-sA} B(s)\, ds\right)' =$

$A Y(t) + \underbrace{e^{tA} e^{-tA}}_{I} B(t) = AY + B$ TORNA

ESERCIZIO

$$\begin{cases} x' = x + y + e^t, & x(0) = 0 \\ y' = -x - y + e^{-t}, & y(0) = 0 \end{cases}$$

$$A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \quad B(t) = \begin{pmatrix} e^t \\ e^{-t} \end{pmatrix} \quad Y_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(Y(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}\right)$$

TROVARE e^{tA} . Polinomio caratteristico

$$\det\begin{pmatrix} 1-\lambda & 1 \\ -1 & -1-\lambda \end{pmatrix} = \lambda^2 - 1 + 1 = \lambda^2 \Rightarrow \lambda_1 = 0 \quad m_1 = 2$$

$B = A - 0I = A \qquad A \neq 0 \qquad \ker A \neq \mathbb{R}^2$

(rango $A = 1$) (c'è un solo autovalore)

$A = B \begin{matrix} e_2 \\ \downarrow \\ e_1 \\ \downarrow \\ 0 \end{matrix}$ Trovo e_2 prendendo un qualunque vettore su $Ae_2 \neq 0$

per esempio $e_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad Be_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = e_1$

$$M = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \quad ([e_1, e_2]) \qquad M^{-1} = \frac{1}{\det M}\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\left[M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad M^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \right]$$

ALLORA $J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad e^{tJ} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

$$e^{tA} = M e^{tJ} M^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} t & -1+t \\ 1 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1+t & t \\ -t & 1-t \end{pmatrix}$$

QUINDI $Y(t) = e^{tA}\left(0 + \int_0^t e^{-sA} B(s)\right) =$

$$e^{tA}\int_0^t \begin{pmatrix} 1-s & -s \\ s & 1+s \end{pmatrix}\begin{pmatrix} e^s \\ e^{-s} \end{pmatrix} ds = e^{tA}\int_0^t \begin{pmatrix} e^s - se^s - se^{-s} \\ se^s + e^{-s} + se^{-s} \end{pmatrix} ds$$

(a) $\int_0^t e^s\,ds = e^t - 1$ (b) $\int_0^t e^{-s}\,ds = -e^{-t} + 1$

(c) $\int_0^t s e^s\,ds = \left[se^s\right]_0^t - \int_0^t e^s\,ds = te^t - e^t + 1$ $\Rightarrow$

(d) $\int_0^t s e^{-s}\,ds = \left[-se^{-s}\right]_0^t + \int_0^t e^{-s}\,ds = -te^{-t} - e^{-t} + 1$

$$Y(t) = e^{tA}\begin{pmatrix} e^t - 1 - te^t + e^t - 1 + te^{-t} + e^{-t} - 1 \\ te^t - e^t + 1 - e^{-t} + 1 - te^{-t} - e^{-t} + 1 \end{pmatrix} \overset{e^{tA}}{=} \begin{pmatrix} 2e^t - t(e^t - e^{-t}) + e^{-t} - 3 \\ -2e^{-t} + t(e^t - e^{-t}) - e^t + 3 \end{pmatrix}$$

$$= \begin{pmatrix} t+1 & t \\ -t & 1-t \end{pmatrix}\begin{pmatrix} 2e^t - t(e^t - e^{-t}) + e^{-t} - 3 \\ -2e^{-t} + t(e^t - e^{-t}) - e^t + 3 \end{pmatrix} =$$

$$\begin{pmatrix} (t+1)(2e^t - t(e^t - e^{-t}) + e^{-t} - 3) + t(-2e^{-t} + t(e^t - e^{-t}) - e^t + 3) \\ -t(2e^t - t(e^t - e^{-t}) + e^{-t} - 3) + (1-t)(-2e^{-t} + t(e^t - e^{-t}) - e^t + 3) \end{pmatrix}$$

$= \ldots$ (CONCLUDERE E VEDERE SE TORNA)

$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ autovalore $\Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e_1$

$e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $Je_2 = \lambda e_2 + e_1$

A 5×5 un solo λ $B = A - \lambda I$

$e_5 \to e_3 \to e_1$ $\quad e_4 \to e_2$

$m_G = 2$ e_1 e_2 gli autovettori

Se prendo $M = (e_1\ e_3\ e_5 \mid e_2\ e_4) \rightsquigarrow J = \left(\begin{array}{ccc|cc} \lambda & 1 & & & \\ & \lambda & & & \\ & & \lambda & & \\ \hline & & & \lambda & 1 \\ & & & & \lambda \end{array}\right)$

Se prendo $M = (e_2\ e_4 \mid e_1\ e_3\ e_5) \rightsquigarrow J = \left(\begin{array}{cc|ccc} \lambda & 1 & & & \\ & \lambda & & & \\ \hline & & \lambda & 1 & \\ & & & \lambda & 1 \\ & & & & \lambda \end{array}\right)$

e^A NON CAMBIA (CAMBIANO I PASSAGGI)

$$A = \begin{bmatrix} 3 & 4 & -2 \\ 2 & 1 & -4 \\ 1 & 2 & 0 \end{bmatrix} \qquad Y' = AY \qquad Y(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

polinomio caratteristico

$$\det \begin{pmatrix} 3-\lambda & 4 & -2 \\ 2 & 1-\lambda & -4 \\ 1 & 2 & -\lambda \end{pmatrix} = -\lambda(3-\lambda)(1-\lambda) - 16 - 8 + 2(1-\lambda) + 8(3-\lambda) + 8\lambda =$$

$-\lambda(\lambda^2 - 4\lambda + 3) - 24 + 2 - 2\lambda + 24 - 8\lambda + 8\lambda =$

$-\lambda^3 + 4\lambda^2 - 3\lambda - 2\lambda + 2 = -\lambda^3 + 4\lambda^2 - 5\lambda + 2$ provo $\lambda = \pm 1$ ..

$\lambda = 1$ È SOLUZIONE !! DIVIDO PER $(\lambda - 1)$

	-1	4	-5	$+2$
1		-1	3	-2
	-1	3	-2	0

$\rightsquigarrow -\lambda^2 + 3\lambda - 2$ RADICI $\frac{-3 \pm \sqrt{9-8}}{-2} = \frac{3 \pm 1}{2} \begin{cases} 1 \\ 2 \end{cases}$

$\lambda_1 = 1$ DOPPIA $\lambda_2 = 2$ $\quad J = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 2 \end{pmatrix}$

$B_1 = A - \lambda_1 I = \begin{pmatrix} 2 & 4 & -2 \\ 2 & 0 & -4 \\ 1 & 2 & -1 \end{pmatrix}$ ← RANGO ?? $= 2$ $\quad \dim(\ker(B_1)) = 1$

$\dim \ker B_1^2 = 2 \quad (\ker B_1^n = \ker B_1^2 \ \forall n \geq 2)$

$\lambda_1 = 1$

$e_3 \quad \lambda_2 = 2$

$B_1 \downarrow$

$e_2 \quad e_1$

$B_1 \downarrow \quad \downarrow B_2$

$0 \quad 0$

TROVO e_3 prendendo e_3 tale che $B_1^2 e_3 = 0 \quad B_1 e_3 \neq 0$ $\quad \left(= e_2\right)$

$B^2 = \begin{bmatrix} 10 & 4 & -18 \\ 0 & 0 & 0 \\ 5 & 2 & -9 \end{bmatrix}$ (fidarsi dei calcoli) (rango 1)

$e = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad 5x + 2y - 9z = 0$ $\quad$ posso a meno che $z = 0 \quad x = -2 \quad y = 5$

$\underbrace{\begin{bmatrix} 2 & 4 & -2 \\ 2 & 0 & -4 \\ 1 & 2 & -1 \end{bmatrix}}_{A - I} \begin{pmatrix} -2 \\ 5 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 + 20 \\ -4 \\ -2 + 10 \end{pmatrix} = \begin{pmatrix} 16 \\ -4 \\ 8 \end{pmatrix} \neq 0 \leftarrow e_2$

ADESSO DEVO TROVARE e_1 in $\ker(A - 2I)$

$A - 2I = \begin{bmatrix} 1 & 4 & -2 \\ 2 & -1 & -4 \\ 1 & 2 & -2 \end{bmatrix}$ $\quad \left(e_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right)$ $\quad \uparrow z = 1$

$\downarrow$ RIDUZIONE

$\begin{bmatrix} 0 & 2 & 0 \\ 0 & -5 & 0 \\ 1 & 2 & -2 \end{bmatrix}$ $\quad \text{I}^\circ - \text{III}^\circ$ $\quad \text{II}^\circ - 2\,\text{III}^\circ$

$\leftrightarrow \begin{bmatrix} 1 & 2 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$ $\quad y = 0 \quad x - 2z = 0$

$$M = \left(\begin{array}{c|cc} e_1 & e_2 & e_3 \end{array}\right) \qquad J = \left(\begin{array}{c|cc} 2 & 0 & 0 \\ \hline 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right)$$

"

$$M = \begin{bmatrix} 2 & 16 & -2 \\ 0 & -4 & 5 \\ 1 & 8 & 0 \end{bmatrix}$$

(AVREI POTUTO METTERE $M = \left(\begin{array}{cc|c} e_2 & e_3 & e_1 \end{array}\right)$ $J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$)

$$M^{-1} = \frac{1}{8}\begin{bmatrix} 40 & 16 & -72 \\ -5 & -2 & 10 \\ -4 & 0 & 8 \end{bmatrix}$$

(fidiamoci)

$$e^{tA} = M \left(\begin{array}{c|cc} e^{2t} & 0 & 0 \\ \hline 0 & e^t & te^t \\ 0 & 0 & e^t \end{array}\right) M^{-1}$$

$$e^t \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$(x(0)=0 \quad y(0)=1 \quad z(0)=0)$

Se voglio la sol. $\rightsquigarrow Y(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow$

$$e^{tA}\begin{pmatrix} 1 \\ 8 \\ 0 \end{pmatrix} = \frac{1}{8} M \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \begin{pmatrix} 16 \\ -2 \\ 0 \end{pmatrix} = \frac{1}{4} M \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{pmatrix} \begin{pmatrix} 8 \\ -1 \\ 0 \end{pmatrix} =$$

se nello $t=0$ $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ ok

$$\frac{1}{4}\begin{pmatrix} 2 & 16 & -2 \\ 0 & -4 & 5 \\ 1 & 8 & 0 \end{pmatrix}\begin{pmatrix} 8e^{2t} \\ -e^t \\ 0 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 16e^{2t} - 16e^t \\ 4e^t \\ 8e^{2t} - 8e^t \end{pmatrix} = \begin{pmatrix} 4e^{2t} - 4e^t \\ e^t \\ 2e^{2t} - 2e^t \end{pmatrix}$$

PROVIAMO A FARE LA VERIFICA dell'eq.

$$x' = 8e^{2t} - 4e^t$$
$$y' = e^t$$
$$z' = 4e^{2t} - 2e^t$$

$3x + 4y - 2z = 12e^{2t} - 12e^t + 4e^t - 4e^t + 4e^t = 8e^{2t} - 4e^t$ ok

COMPLETARE ..

$$y_1' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} y$$

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad P(\lambda) = (1-\lambda)^2 + 1$$

$1 - \lambda = \pm i \qquad \lambda = 1 \pm i$ (COMPLESSO CONIUGATO)

SEMPLICI

$$J = \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}$$ CERCHIAMO LA DIAGONALIZZAZIONE IN $\mathbb{C}$

$$B_1 = A - \lambda_1 I = A - (1+i) I = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \qquad e_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$e_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$ che è ortogonale a e_1 (Prodotto scalare è $e \cdot \bar{e}$)

$$e_1 \cdot e_2 = 1 \cdot \bar{i} + i \cdot \bar{1} = -i + i = 0$$

oppure prendete ev nel ker di $\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}$

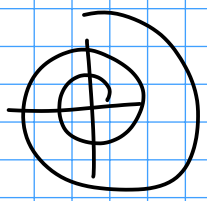
$$M = \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix} \qquad M^{-1} = \frac{1}{-2}\begin{pmatrix} i & -1 \\ -1 & i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix}$$

$$e^{tA} = \frac{1}{2}\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}\begin{pmatrix} e^t e^{it} & 0 \\ 0 & e^t e^{-it} \end{pmatrix}\begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix} =$$

$$\frac{e^t}{2}\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}\begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix} =$$

$$\frac{e^t}{2}\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}\begin{pmatrix} -i e^{it} & e^{it} \\ e^{-it} & -i e^{-it} \end{pmatrix} = \frac{e^t}{2}\begin{pmatrix} e^{it} + e^{-it} & i(e^{it} - e^{-it}) \\ -i(e^{it} - e^{-it}) & e^{it} + e^{-it} \end{pmatrix} =$$

$$e^t \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} = e^{tA}$$



$$\begin{cases} x' = y \\ y' = x + z \\ z' = y \end{cases} \qquad A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \begin{matrix} x(0) = 1 \\ y(0) = \sqrt{2} \\ z(0) = 1 \end{matrix}$$

$$P(\lambda) = \det \begin{pmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{pmatrix} = (-1)\cdot(-\lambda) + (-\lambda)\det \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} =$$

$$\lambda - \lambda(\lambda^2 - 1) = -\lambda^3 + 2\lambda = -\lambda(\lambda^2 - 2)$$

autovalori: $0, \sqrt{2}, -\sqrt{2}$ tutti semplici SI DIAGONALIZZA

Devo trovare gli autovettori

$\lambda = 0$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} x + z = 0 \\ y = 0 \end{matrix}$$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$\lambda = \sqrt{2}$

$$\begin{bmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{bmatrix} \rightsquigarrow \begin{bmatrix} -\sqrt{2} & 1 & 0 \\ 0 & -1 & \sqrt{2} \\ 0 & 1 & -\sqrt{2} \end{bmatrix} \qquad \begin{matrix} -\sqrt{2}x + y = 0 & y = \sqrt{2}x \\ -y + \sqrt{2}z = 0 & y = \sqrt{2}z \end{matrix}$$

$$e_2 = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

$\lambda = -\sqrt{2}$

$$\begin{bmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{bmatrix} \rightsquigarrow \begin{bmatrix} \sqrt{2} & 1 & 0 \\ 0 & -1 & -\sqrt{2} \\ 0 & 1 & \sqrt{2} \end{bmatrix} \qquad \begin{matrix} y = -\sqrt{2}x \\ y = -\sqrt{2}z \end{matrix}$$

$$e_3 = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \qquad M = \begin{pmatrix} 1 & 1 & 1 \\ 0 & \sqrt{2} & -\sqrt{2} \\ -1 & 1 & 1 \end{pmatrix} \qquad J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & -\sqrt{2} \end{pmatrix}$$

$$e^{tA} = M \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\sqrt{2}t} & 0 \\ 0 & 0 & e^{-\sqrt{2}t} \end{pmatrix} M^{-1}$$

COND. INIZIALE $\qquad Y(0) = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} = 2e_1$

so che $M\hat{e}_1 = e_1 \iff M^{-1}e_1 = \hat{e}_1$ (ma calcolo M^{-1} ...)

$$e^{tA} Y_0 = M e^{tJ} 2\hat{e}_2 = 2M \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\sqrt{2}it} & 0 \\ 0 & 0 & e^{-\sqrt{2}it} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = M \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= 2e_1 = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \qquad Y(t) \text{ è COSTANTE !!}$$

PROVATE NELLO STESSO MODO A RISOLVERE CON DATO INIZIALE

$$Y_0 = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$