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Lezioni di Analisi Matematica 2 e Complementi

Lezione 61 16/5/18

(*) ricevimento il mercoledì alle 15.00 al Dip. Matematica
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$$\vec{f}(x,y,z) = y(z+3y^2)\,\vec{i} + x(z-3x^2)\,\vec{j} + \frac{x^2y^2}{1+x^2+y^2}\,\vec{k}$$

$$\vec{F}(x,y,z) = \left(\frac{xz^2}{2} - 3x^3z\right)\vec{i} +$$

$$\left(-y\frac{z^2}{2} - 3y^3z + xy^2 - y^2\sqrt{1+y^2}\arctan\left(\frac{x}{1+y^2}\right)\right)\vec{j}$$

$\vec{F}$ è un potenziale vettore per $\vec{f}$ cioè $\operatorname{rot}\vec{F} = \vec{f}$ $(\operatorname{div}\vec{f} = 0)$

Prendiamo $S = \{(x,y,z) : x^2+y^2 \le 1 \quad z = 4x^2+y^2\}$

Voglio calcolare il flusso di $\vec{f}$ attraverso S con normale $\hat{\nu}$ concorde con l'asse z.

$$\iint_S \vec{f}\cdot\hat{\nu}\,d\sigma = ?!?$$

(A) Uso la def. di integrale di superficie. Devo parametrizzare S:

S è il grafico di $\left(z = \underbrace{g(x,y) = 4x^2+y^2}\right)$ su $B = \{x^2+y^2 \le 1\}$

Dunque S è una sup. parametrica e

$$\vec{N}(x,y) = -\frac{\partial g}{\partial x}\,\vec{i} - \frac{\partial g}{\partial y}\,\vec{j} + \vec{k} = -8x\,\vec{i} - 2y\,\vec{j} + \vec{k}$$

che è concorde con $\vec{k}$ (e dunque va bene per calcolare il flusso)

$$\iint_S \vec{f}\cdot\hat{v}\,d\sigma = \iint_B \vec{f}(x,y,g(x,y))\cdot\vec{N}(x,y)\,dx\,dy =$$

$$\iint_B \left\{\left[y\left(4x^2+y^2+3y^2\right)\right]\vec{i} + \left[x\left(4x^2+y^2-3x^2\right)\right]\vec{j} + \frac{x^2y^2}{1+x^2+y^2}\vec{k}\right\}\cdot\left(-8x\vec{i}-2y\vec{j}+\vec{k}\right)dx\,dy$$

$$= \iint_B \Big[\underbrace{-8xy\left(4x^2+4y^2\right)}_{=0} \underbrace{-2xy\left(x^2+y^2\right)}_{=0} + \frac{x^2y^2}{1+x^2+y^2}\Big]\,dx\,dy \qquad \text{COORD. POLARI}$$

per simmetria rispetto a un asse

$$\int_0^{2\pi} d\theta \int_0^1 \rho\,d\rho \left[-8\sin\theta\cos\theta\rho\cdot 4\rho^2 - 2\rho\cos\theta\,\rho\sin\theta\,\rho^2 + \frac{\rho^2\cos^2\theta\,\rho^2\sin^2\theta}{1+\rho^2}\right] =$$

$$-32\int_0^{2\pi}\sin\theta\cos\theta\int_0^1\rho^5\,d\rho - 2\int_0^{2\pi}\sin\theta\cos\theta\,d\theta\int_0^1\rho^3\,d\rho + \int_0^{2\pi}\cos^2\theta\sin^2\theta\,d\theta\int_0^1\frac{\rho^5}{1+\rho^2}\,d\rho$$

$$\int_0^{2\pi}\sin\theta\cos\theta\,d\theta = \frac{1}{2}\int_0^{2\pi}\sin 2\theta\,d\theta = \frac{1}{2}\left[\frac{-\cos 2\theta}{2\theta}\right]_0^{2\pi} = 0$$

RIMANE L'ULTIMO INTEGRALE:

$$\int_0^{2\pi}\cos^2\theta\sin^2\theta\,d\theta = \frac{1}{4}\int_0^{2\pi}\sin(2\theta)^2\,d\theta = \frac{1}{4}\int_0^{2\pi}\frac{1-\cos(4\theta)}{2}\,d\theta = \frac{2\pi}{8} = \frac{\pi}{4}$$

$$\int_0^1 \frac{\rho^5}{1+\rho^2}\,d\rho \qquad (s=\rho^2 \quad ds=2\rho\,d\rho) = \frac{1}{2}\int_0^1\frac{s^2}{1+s}\,ds =$$

$$\frac{1}{2}\int_0^1\left(\frac{s^2-1}{s+1}+\frac{1}{s+1}\right)ds = \frac{1}{2}\int_0^1\left(s-1+\frac{1}{s+1}\right)ds =$$

$$\frac{1}{2}\left[\frac{s^2}{2}-s+\ln(1+s)\right]_0^1 = \frac{1}{2}\left(\frac{1}{2}-1+\ln(2)\right) = \frac{2\ln(2)-1}{4}$$

$$\Rightarrow \text{FLUSSO} = \frac{2\ln(2)-1}{16}\pi$$

PROVIAMO A CALCOLARE LO STESSO FLUSSO USANDO STOKES

dato che sappiamo che $\vec{f} = \operatorname{rot}\vec{F}$ DUNQUE

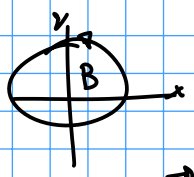
$$\iint_S \vec{f}\cdot\hat{v}\,d\sigma = \iint_S \operatorname{rot}\vec{F}\cdot\hat{v}\,d\sigma = \int_{\partial\Sigma(S)}\vec{F}\cdot d\vec{s}$$

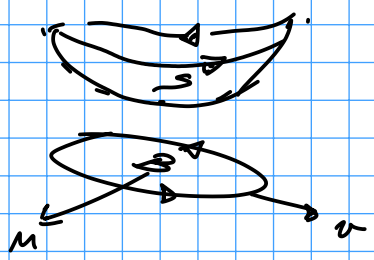
dove $\Sigma(S)$ deve essere orientato coerentemente con $\hat{v}$

Per farlo considero una curva $\gamma(t)$

che descrive ∂B coerente con B:

$$\gamma(\theta) = \cos\theta\,\hat{u} + \sin\theta\,\hat{v} \qquad 0 \le \theta \le 2\pi$$





e cosicché $\tilde{\gamma}(\theta) = \cos\theta\,\vec{\imath} + \sin\theta\,\vec{\jmath} + g(\cos\theta, \sin\theta)\,\vec{k} =$

$$\cos\theta\,\vec{\imath} + \sin\theta\,\vec{\jmath} + \left(4\cos^2\theta + \sin^2\theta\right)\vec{k} = \qquad 0 \le \theta \le 2\pi$$

$\tilde{\gamma}(\theta)$ $\quad \cos\theta\,\vec{\imath} + \sin\theta\,\vec{\jmath} + \left(3\cos^2\theta + 1\right)\vec{k}$ ← descrive $\Sigma(S)$

$$\tilde{\gamma}'(\theta) = -\sin\theta\,\vec{\imath} + \cos\theta\,\vec{\jmath} - 6\cos\theta\sin\theta\,\vec{k}$$

Calcolo l'integrale di linea usando la definizione:

$$\int_0^{2\pi} \vec{F}(\tilde{\gamma}(\theta)) \cdot \tilde{\gamma}'(\theta)\, d\theta =$$

$$\int_0^{\pi} \Big\{ \frac{\cos\theta}{2} \underbrace{\left(3\cos^2\theta + 1\right)^2}_{s=\cos\theta} (-\sin\theta) - 3\cos^3\theta\left(3\cos^2\theta + 1\right)(-\sin\theta) +$$

$$\Big[-\frac{\sin\theta}{2} \underbrace{\left(3\cos^2\theta + 1\right)^2}_{s=\sin\theta} - 3\sin^3\theta\left(3\cos^2\theta + 1\right) + \cos\theta\sin^2\theta$$

$$- \sin^2\theta\sqrt{1+\sin^2\theta}\,\operatorname{arctan}\left(\frac{\cos\theta}{1+\sin^2\theta}\right)\Big]\cos\theta \Big\}\, d\theta \qquad (F_3 = 0)$$

$$= \int_1^2 \frac{s}{2}\left(3s^2+1\right)^2 ds + \int_1^0 -3s^3\left(3s^2+1\right) ds + \int_0^0 -\frac{s}{2}\left(4-s^2\right)^2 - 3s^3(4-s)$$

$$= 0 \qquad\qquad = 0 \qquad\qquad = 0$$

$$+ \int_0^{2\pi} \cos^2\sin^2\vartheta\, d\theta - \int \sin^2\theta\sqrt{1+\sin^2\theta}\,\operatorname{arctan}\left(\frac{\cos\theta}{1+\sin^2\theta}\right)\cos\theta\, d\theta$$

$$= \frac{\pi}{4} \qquad\qquad = \text{??}$$

LA SOMMA DEI DUE FA $\frac{2\Theta(1)-1}{16}\pi$

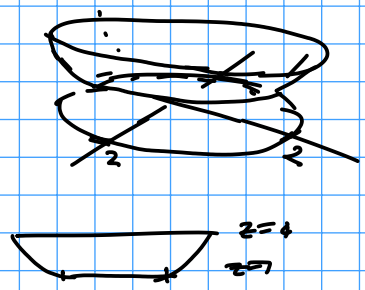
Da Stokes ricavo il valore dell'ultimo integrale..

ESERCIZIO Doh

$$\Omega = \{x^2 + y^2 \le 4, \quad 1 \le z \le x^2 + y^2\}$$

$$S = \{1 \le x^2 + y^2 \le 4, \quad z = x^2 + y^2$$

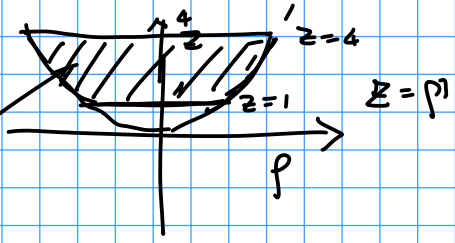
$$\vec{f}(x, y, z) = \frac{2y}{x^2 + y^2}\,\vec{i} - \frac{3x}{x^2 + y^2}\,\vec{j} - \vec{k}$$



Calcolare il flusso di $\vec{f}$ attraverso:

(1) $\partial\Omega$ (2) S

LA PROSSIMA VOLTA



Ω è ottenuto ruotando