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Lezioni di Analisi Matematica 2 e Complementi

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Teorema (della divergenza)

Supponiamo $D \subset \mathbb{R}^3$ un dominio regolare ($D = \{G(x) \le 0\}$ dove $G: \mathbb{R}^3 \to \mathbb{R}$ di classe C^1, $\nabla G(x) \neq 0 \ \forall x \in \partial D = \{G(x) = 0\}$ – o più D regolare a tratti – Supponiamo $\vec{f}: D \to \mathbb{R}^3$ di classe C^1. Allora

$$\iiint_D \operatorname{div} \vec{f}(x,y,z)\, dx\,dy\,dz = \iint_{\partial D} \vec{f} \cdot \hat{\nu}\, d\sigma$$

dove su ∂D (è una superficie regolare o ~~tratti~~) consideriamo la normale uscente da D ($\Rightarrow \partial D$ è orientabile!)

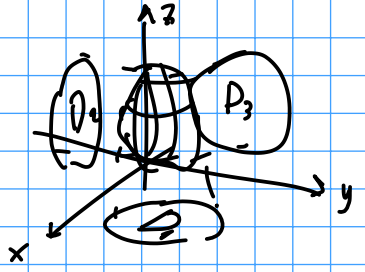
Dim. (Caso particolare) Supponiamo (in aggiunta) che D sia un insieme normale rispetto a ognuno degli assi.

$$D = \{(x,y,z) : (x,y) \in D_1 \ \ g_1(x,y) \le z \le h_1(x,y)\} =$$

(II) $$= \{(x,y,z) : (x,z) \in D_2 \ \ g_2(x,y) \le y \le h_2(x,y)\} =$$

(III) $$= \{(x,y,z) : (y,z) \in D_3 \ \ g_3(x,y) \le x \le h_3(x,y)\}$$

con g_i, h_i di classe C^1, $i = 1,2,3$



(per esempio $D = \{x^2+y^2+z^2 \le 1\}$ è OK

$$\vec{f}(x,y,z) = f_1(x,y,z)\,\vec{i} + f_2(x,y,z)\,\vec{j} + f_3(x,y,z)\,\vec{k}$$

$$\operatorname{div}\vec{f} = \frac{\partial}{\partial x} f_1 + \frac{\partial}{\partial y} f_2 + \frac{\partial}{\partial z} f_3$$

$$\iiint_D \frac{\partial f_3}{\partial z}\,dx\,dy\,dz = \iint_{D_1} dx\,dy \int_{g_1(x,y)}^{R_1(x,y)} \frac{\partial f_3}{\partial z}\,dz =$$

$$\iint_{D_1} f_3(R_1(x,y))\,dx\,dy - \iint_{D_1} f_3(g_1(x,y))\,dx\,dy \quad (=)$$

$\iint_{\Sigma_1}$ $\qquad$ $\iint_{\Sigma_3}$

$$\iint_{\partial D} (f_3 \cdot \vec{k}) \cdot \hat{\nu}\,d\sigma \qquad ??$$

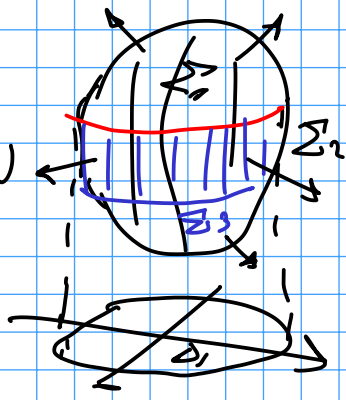
IN EFFETTI (VOGLIO VERIFICARE ??) posso scrivere

$$\partial D = \{(x,y,z) : (x,y) \in D_1 \quad z = R_1(x,y)\} \cup$$

$$(\Sigma_1)\ \{(x,y,z) : (x,y) \in \partial D_1 \quad g_1(x,y) \le z \le R_1(x,y)\} \cup$$

$$\{(x,y,z) : (x,y) \in D_1 \quad z = g_1(x,y)\}$$

Σ_3



VEDIAMO CHE

$$\iint_{D_1} f_3(R_1(x,y))\,dx\,dy = \iint_{\Sigma_1} (f_3 \vec{k}) \cdot \hat{\nu}\,d\sigma$$

infatti Σ_1 è un grafico, la normale a Σ_1 è

$$\vec{N}(x,y) = \left(-\frac{\partial}{\partial x} R_1,\ -\frac{\partial}{\partial y} R_1,\ 1\right) \qquad \text{ADES} \quad \vec{N}(x,y) \cdot \vec{k} = 1$$

Analogamente vedo che $\iint_{D_1} f_3(g_1(x,y))\,dx\,dy = -\iint_{\Sigma_3} (f_3 \vec{k}) \hat{\nu}\,d\sigma$

(stavolta $\vec{N}(x,y) \cdot \hat{\nu} = -1$)

INVECE IL TRATTO VERTICALE DÀ CONTRIBUTO NULLO

$$\iint_{\Sigma_2} (f_3 \cdot \vec{k})\, \hat{\nu}\, d\sigma \Rightarrow \quad \text{perché } \hat{\nu} \cdot \vec{k} \Rightarrow 0 \quad \text{nei punti di } \Sigma_2$$

DUNQUE

$$\iiint_D \frac{\partial}{\partial z} f_3 \, dx\,dy\,dz = \iint_{\partial D} (f_3 \vec{k}) \cdot \hat{\nu}\, d\sigma$$

ANALOGAMENTE TROVO

$$\iiint_D \frac{\partial}{\partial y} f_2 \, dx\,dy\,dz = \iint_{\partial D} (f_2 \cdot \vec{j}) \cdot \hat{\nu}\, d\sigma$$

(uso la decomposizione rispetto all'asse y (II))

$$\iiint_D \frac{\partial}{\partial x} f_1 \, dx\,dy\,dz = \iint_{\partial D} (f_1 \cdot \vec{i}) \cdot \hat{\nu}\, d\sigma$$

(uso la decomposizione rispetto all'asse x (III))

SE SOMMO TROVO IL TEOREMA

$$\iiint_D \operatorname{div} \vec{f}\, dx\,dy\,dz = \iint \underbrace{(f_1 \vec{i} + f_2 \vec{j} + f_3 \vec{k})}_{\vec{f}} \cdot \hat{\nu}\, d\sigma$$

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ESEMPI $\quad \vec{f} = 2x\vec{i} - xy\vec{j} + \vec{k}$

$D = \{ x^2 + y^2 + z^2 \le 1 \}$

Proviamo a calcolare $\iint_{\partial D} \vec{f} \cdot \hat{\nu}\, d\sigma$ ($\hat{\nu}$ = NORMALE USCENTE DA D)

Descrivo $\partial D = \{x^2 + y^2 + z^2 = 1\}$ mediante le coordinate sferiche

$$x = \cos\theta \sin\varphi \qquad y = \sin\theta \sin\varphi \qquad z = \cos\varphi$$

$$\Gamma_1 \qquad\qquad \Gamma_2 \qquad\qquad \Gamma_3$$

$$\frac{\partial \Gamma}{\partial \theta} = \begin{pmatrix} -\sin\theta \sin\varphi \\ \cos\theta \sin\varphi \\ 0 \end{pmatrix} \qquad \frac{\partial \Gamma}{\partial \varphi} = \begin{pmatrix} \cos\theta \cos\varphi \\ \sin\theta \cos\varphi \\ -\sin\varphi \end{pmatrix}$$

$$\frac{\partial \Gamma}{\partial \varphi} \otimes \frac{\partial \Gamma}{\partial \theta} = \det \begin{bmatrix} i & \cos\theta\cos\varphi & -\sin\theta\sin\varphi \\ j & \sin\theta\cos\varphi & \cos\theta\sin\varphi \\ k & -\sin\varphi & 0 \end{bmatrix} =$$

in evidenza

$$\sin\theta \det \begin{bmatrix} i & \cos\theta\cos\varphi & -\sin\theta \\ j & \sin\theta\cos\varphi & \cos\theta \\ k & -\sin\varphi & 0 \end{bmatrix} =$$

$$\sin\varphi \left[-\cos\theta\sin\varphi\, \vec{i} - \sin\theta\sin\varphi\, \vec{j} - \left(\cos\varphi(\cos^2\theta+\sin^2\theta)\right)\vec{k} \right] =$$

$$-\sin\varphi\ \Gamma(\theta,\varphi)$$

(verso sbagliato!)

$$\Gamma = \Gamma(\theta,\varphi) \Rightarrow \frac{\partial \Gamma}{\partial \theta} \otimes \frac{\partial \Gamma}{\partial \varphi} = \sin\varphi\ \Gamma(\theta,\varphi) = \vec{N}(\theta,\varphi)$$

PUNTA VERSO FUORI ($\sin\varphi \geq 0$)

$$\Rightarrow \iint_{\partial D} \vec{f}\cdot\hat{\nu}\, d\sigma =$$

(Noto che usato le coord. sferiche anche se il dominio non è "semplice")

$$\int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \left(\vec{f}(\Gamma(\theta,\varphi)) \cdot \vec{N}(\theta,\varphi) \right) =$$

$$\int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \Big(\underbrace{2\cos^2\theta\sin\varphi\cdot\cos\theta\sin\varphi}_{f_1\cdot\nu_1} - \underbrace{\cos\theta\sin\varphi\sin\theta\sin\varphi\sin\theta\sin\varphi}_{f_2\nu_2} + \underbrace{\cos\varphi}_{f_3\nu_3} \Big) \sin\varphi$$

$$\int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \left(2\cos^2\theta\sin^2\varphi - \cos\theta\sin^2\theta\sin^4\varphi + \sin\varphi\cos\varphi \right) =$$

$$2\int_0^{2\pi}\cos^2\theta\, d\theta \int_0^{\pi}\sin^2\varphi\, d\varphi - \int_0^{2\pi}\cos\theta\sin^2\theta\, d\theta \int_0^{\pi}\sin^4\varphi\, d\varphi + 2\pi\int_0^{\pi}\sin\varphi\cos\varphi\, d\varphi$$

① ② ③

$$③ = \pi\int_0^{\pi}\sin 2\varphi\, d\varphi = 0$$

in ②: $\int_0^{2\pi}\cos\theta\sin^2\theta\, d\theta =$ $t=\sin\theta$, $dt=\cos\theta\, d\theta$ →

$$\int_0^0 t^2\, dt = 0 \Rightarrow ② = 0$$

$$① \rightarrow \int_0^{\pi}\sin^3\varphi\, d\varphi = \int_0^{\pi}(1-\cos^2\varphi)\sin\varphi\, d\varphi \qquad t=\cos\varphi,\ dt=-\sin\varphi\, d\varphi$$

$$= \int_1^{-1}(1-t^2)(-dt) = \int_{-1}^{1}(1-t^2)\,dt = \left[t-\frac{t^3}{3}\right]_{-1}^{1} = 2\left(1-\frac{1}{3}\right) = \frac{4}{3}$$

$$2\int_0^{2\pi} \cos^2\theta \, d\theta = \int_0^{2\pi} \frac{\cos(2\theta)+1}{2}\, dt = 2\pi \cdot \frac{1}{2} = \pi$$

ALLA FINE $\displaystyle\iint_{\partial D} \vec{f}\cdot\hat{\nu}\, d\sigma = \frac{4}{3}\pi$ $\quad \frac{8}{3}\pi$

Se calcolo la divergenza trovo $\operatorname{div}\vec{f} = 2 - x$

Devo integrare $2-x$ su D $\rightarrow$

$$\iiint_{\{x^2+y^2+z^2\le 1\}} (2-x)\,dx\,dy\,dz \qquad \left(\text{dovrebbe fare } \frac{4}{3}\pi\right) \ldots$$

$$\parallel$$

$$\int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^1 (2-\rho\cos\theta\sin\varphi)\sin\varphi\, \rho^2\, d\rho = \ldots$$

$$\int_0^{2\pi} d\theta \int_0^{\pi} 2\sin\varphi \int_0^1 \rho^2\, d\rho - \int_0^{2\pi}\cos\theta\, d\theta \underbrace{\int_0^{\pi}\sin\varphi\, d\varphi}_{=0} \int_0^1 \rho^3\, d\rho$$

$$\parallel$$

$$2\pi \left[-2\cos\varphi\right]_0^{\pi} \frac{\rho^3}{3}\Big|_0^1 = \frac{8}{3}\pi$$

AVEVO SCORDATO UN 2 PRIMA