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Lezioni di Analisi Matematica 2 e Complementi

Lezione 34 13/12/17

(*) ricevimento il venerdì alle 10.30 al Dip. Matematica
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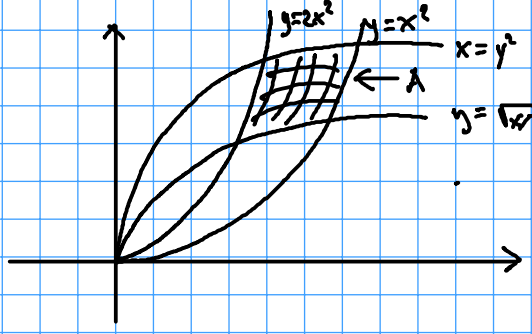
ESERCIZIO Calcolare l'area:

$|A|$ dove $A = \{(x,y) \in \mathbb{R}^2 : x^2 \le y \le 2x^2,\ y^2 \le x \le 2y^2\}$



Potrei vedere A come unione di tre pezzi normali e usare Fubini...

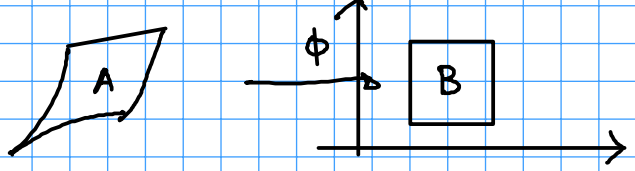
(LUNGO) Posso cercare di trasformare A in un'insieme più semplice.

$\phi(x,y) := \left(\frac{y}{x^2}, \frac{x}{y^2}\right)$ ($\frac{y}{x^2} = v$, $\frac{x}{y^2} = s$)

$\phi : Q_1 \to \mathbb{R}^2$

$Q_1 = \{x>0, y>0\}$

ϕ trasforma A nell'insieme $B = \{(r,s) : 1 \le r \le 2,\ 1 \le s \le 2\}$



VORREI APPLICARE UN TEOR. DI CAMBIO DI VARIABILE. Se applico il teorema a ϕ devo ($\beta(r,s) = \begin{cases} 1 & \text{se } (r,s) \in B \\ 0 & \text{altrimenti} \end{cases}$

$$\iint f \circ \phi \left| \det \frac{\partial \phi}{\partial(x,y)} \right| = |B| \quad (= 1)$$

$$\iint_{\phi^{-1}(B)} \left| \det \frac{\partial \phi}{\partial(x,y)} \right| dx\,dy = \iint_A \left| \det \frac{\partial \phi}{\partial(x,y)} \right| dx\,dy$$

NON MI ILLUMINA SU $|B|$?!

Devo applicare il teorema a $\Phi^{-1} = \Psi$; dunque devo calcolare $(x,y) = \Psi(r,s)$; devo cioè risolvere il sistema (non lineare)

$$\begin{cases} r = \frac{y}{x^2} \\ s = \frac{x}{y^2} \end{cases} \iff \begin{cases} y = r x^2 \\ x = s y^2 \end{cases} \iff \begin{cases} y = r s^2 y^4 \\ x = s y^2 \end{cases}$$

(Poiché $r>0$, $s>0$) $\iff$ $\begin{cases} 1 = r s^2 y^3 \\ x = s y^2 \end{cases}$ $\iff$

$$y = \left(r^{-1} s^{-2} \right)^{1/3} = r^{-\frac{1}{3}} s^{-\frac{2}{3}} \qquad (s>0,\ r>0)$$

$$x = s y^2 = s\, r^{-\frac{2}{3}} s^{-\frac{4}{3}} = r^{-\frac{2}{3}} s^{-\frac{1}{3}}$$

dunque $\Psi : B \to A$ (BIGETTIVA) è data da $\Psi(r,s) = \begin{pmatrix} r^{-1/3} s^{-2/3} \\ r^{-2/3} s^{-1/3} \end{pmatrix}$

Calcolo lo Jacobiano di Ψ:

$$\frac{\partial \Psi}{\partial(r,s)} = \begin{pmatrix} -\frac{1}{3} r^{-\frac{4}{3}} s^{-\frac{2}{3}} & , -\frac{2}{3} r^{-\frac{1}{3}} s^{-\frac{5}{3}} \\ -\frac{2}{3} r^{-\frac{5}{3}} s^{-\frac{1}{3}} & , -\frac{1}{3} r^{-\frac{2}{3}} s^{-\frac{4}{3}} \end{pmatrix} \Rightarrow$$

$$\det\left(\frac{\partial \Psi}{\partial(r,s)} \right) = \frac{1}{9} r^{-2} s^{-2} - \frac{4}{9} r^{-2} s^{-2} = -\frac{1}{3} \frac{1}{s^2 r^2}$$

Applico il th. di cambio di variabile :

$$\iint_B \left| \det \frac{\partial \Psi}{\partial(r,s)} \right| dr\,ds = |A|$$

$$\iint_B \frac{1}{3} \frac{dr\,ds}{s^2 r^2} = \frac{1}{3} \int_1^2 \frac{dr}{r^2} \int_1^2 \frac{ds}{s^2} = \frac{1}{3} \left(\int_1^2 \frac{dr}{r^2} \right)^2 =$$

$$\frac{1}{3}\left(\left[-\frac{1}{r} \right]_1^2 \right)^2 = \frac{1}{3}\left(1 - \frac{1}{2} \right)^2 = \boxed{\frac{1}{12}}$$

ESERCIZIO

$$\textcircled{\scriptsize ES} = \int_{\mathbb{R}^2} \frac{dx\,dy}{(1+x^2+y^2)^\alpha} = ?? \qquad \text{con } \alpha \in \mathbb{R}$$

(LA COSA HA SENSO PERCHÉ $\frac{1}{(1+x^2+y^2)^\alpha}$ è continua su $\mathbb{R}$

$\Rightarrow$ è misurabile $\Rightarrow$ HA UN INTEGRALE CHE PUÒ ESSERE $+\infty$.

CONVIENE PASSARE IN COORD. POLARI

$\textcircled{\scriptsize ES}$ DIVENTA $\iint_A \rho \frac{d\rho\,d\theta}{(1+\rho^2)^\alpha}$ $\qquad A = \{(\rho,\theta) \ 0 \le \vartheta \le 2\pi, \ \rho \ge 0\}$

$=$ (TONELLI)

$$\int_0^{2\pi} \left(\int_0^{+\infty} \frac{\rho}{(1+\rho^2)^\alpha} d\rho \right) d\theta = \int_0^{+\infty} \left(\int_0^{2\pi} d\theta \right) \frac{\rho}{(1+\rho^2)^\alpha} d\rho =$$

$$= 2\pi \int_0^{+\infty} \frac{\rho}{(1+\rho^2)^\alpha} d\rho = \quad (s = \rho^2, \ "2\rho\,d\rho = ds") \quad \pi \int_0^{+\infty} \frac{ds}{(1+s)^\alpha} = \pi \lim_{R\to+\infty} \int_0^R \frac{ds}{(1+s)^\alpha}$$

$$\frac{1}{(1+s)^\alpha}$$

$$= \pi \lim_{R\to+\infty} \begin{cases} \left[\frac{(1+s)^{-\alpha+1}}{-\alpha+1} \right]_0^R & \text{se } \alpha \neq 1 \\ \left[\ln(1+s) \right]_0^R & \text{se } \alpha = 1 \end{cases} =$$

$\alpha > 1 \quad (-\alpha+1 < 0)$

$\alpha < 1 \quad (-\alpha+1 > 0)$

$\alpha = 1$

$$\pi \cdot \begin{cases} \frac{1}{\alpha-1} \\ +\infty \\ +\infty \end{cases}$$

$$\begin{cases} \frac{\pi}{\alpha-1} & \text{se } \alpha > 1 \\ +\infty & \text{se } \alpha \le 1 \end{cases}$$

NOTA $\alpha > 1$ corrisponde a $\frac{1}{(1+x^2+y^2)^\alpha} \simeq \frac{1}{\|(x,y)\|^{\beta}}$ con $\beta = 2\alpha > 2$

(in una variabile dovevo: $\int_{\mathbb{R}} \frac{dx}{(1+x^2)^\alpha}$ converge per $\alpha > \frac{1}{2}$

ANALOGAMENTO SI TROVA

$\iint_{B(0,1)} \frac{1}{(x^2+y^2)^\alpha}$ è finito $\Leftrightarrow$ $\alpha < 1$ $\left(2\pi \int_0^1 \frac{\rho\, d\rho}{(\rho^2)^\alpha} \right)$

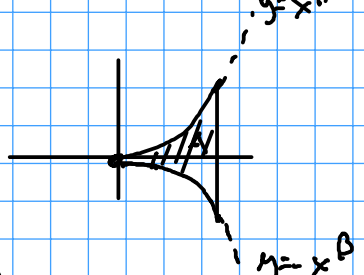
(mentre $\int_0^1 \frac{1}{(x^2)^\alpha}\, dx$ è finito $\Leftrightarrow$ $\alpha < \frac{1}{2}$)

PERÒ questo dipende dal fatto che $B(0,1)$ è proprio un disco.

ALTRO ESEMPIO: (vediamo per quali valori di β è finito l'integrale)

$$\int_A \frac{1}{x^2+y^2}\, dx\, dy = \circledast$$

$A = \{ 0 \le x \le 1 \,,\ |y| \le x^\beta \}$



$$\circledast = \int_0^1 \left(\int_{-x^\beta}^{x^\beta} \frac{dy}{x^2+y^2} \right) dx = 2 \int_0^1 \frac{1}{x^2} \left(\int_0^{x^\beta} \frac{dy}{1+\left(\frac{y}{x}\right)^2} \right) dx$$

$y = sx$ nell'integrale interno $\rightarrow$ $dy = x\, ds$, $\quad 0 \le s \le x^{\beta-1}$

$$2 \int_0^1 \frac{x}{x^2} \int_0^{x^{\beta-1}} \frac{ds}{1+s^2} = 2 \int_0^1 \frac{1}{x} \left[\arctan(s) \right]_0^{x^{\beta-1}} dx =$$

$$2 \int_0^1 \frac{\arctan(x^{\beta-1})}{x}\, dx$$

VOGLIO CAPIRE PER QUALI β è finito.

RICORDO che $\arctan(t) \simeq t$ per $t \sim 0$ $\Rightarrow$

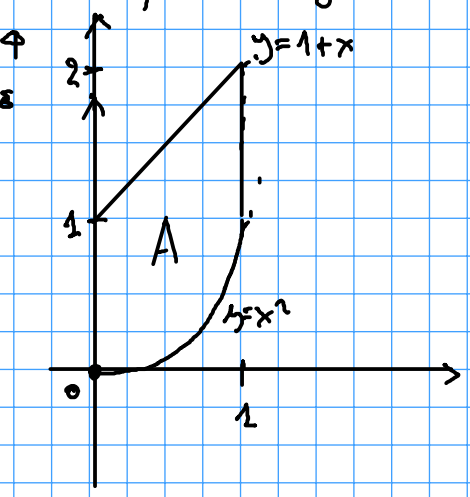
L'INTEGRANDO (≥ 0) ed è $\simeq \frac{x^{\beta-1}}{x} = x^{\beta-2}$ (fuori da zero è continuo)

SO (ANALISI I) che $x^{\beta-2}$ è INTEGRABILE VICINO A ZERO

$\Leftrightarrow$ $\beta - 2 > -1$ ($2 - \beta < 1$) $\Leftrightarrow$ $\beta > 1$

(mentre se A contenesse un intorno di $(0,0)$ $\Rightarrow$ INTEGR. $= +\infty$)

ESERCIZIO $\iint_A xy\,dx\,dy$ dove $A=\{0\le x\le 1,\ x^2\le y\le 1+x\}$



$$= \int_0^1 \left(\int_{x^2}^{1+x} xy\,dy \right) dx = \int_0^1 \left[\frac{xy^2}{2} \right]_{x^2}^{1+x} dx =$$

$$\frac{1}{2}\int_0^1 \left(x(1+x)^2 - x x^4 \right) dx =$$

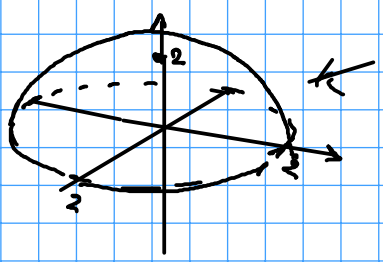
$$\frac{1}{2}\int_0^1 \left(x + 2x^2 + x^3 - x^5 \right) dx =$$

$$\frac{1}{2}\left[\frac{x}{2} + \frac{2}{3}x^3 + \frac{x^4}{4} - \frac{x^6}{6} \right]_0^1 = \frac{1}{2}\left(\frac{1}{2} + \frac{2}{3} + \frac{1}{4} - \frac{1}{6} \right) = \frac{1}{2}\,\frac{6+8+3-2}{12}$$

$$= \frac{1}{2}\,\frac{15}{12} = \frac{5}{8}$$

ESERCIZIO

$$\iiint_A \frac{z}{1+x^2+y^2+z^2}\,dx\,dy\,dz \qquad A=\{x^2+y^2+z^2\le 4,\ z\ge 0\}$$



$$\int_0^2 \left(z \iint_{A_z} \frac{dx\,dy}{1+x^2+y^2+z^2} \right) dz$$

dove A_z ("sezione") $= \{x^2+y^2\le 4-z^2\} \subset \mathbb{R}^2$

$$\iint_{A_z} \frac{dx\,dy}{1+x^2+y^2+z^2} = \frac{1}{1+z^2} \iint_{A_z} \frac{dx\,dy}{1+\left(\frac{x}{\sqrt{1+z^2}}\right)^2+\left(\frac{y}{\sqrt{1+z^2}}\right)^2} = \circledast$$

$t = \frac{x}{\sqrt{1+z^2}}$, $w = \frac{y}{\sqrt{1+z^2}}$ (LINEARE, z è costante per ora)

$x = \sqrt{1+z^2}\,t$, $y = \sqrt{1+z^2}\,w$

$dx\,dy = (1+z^2)\,dt\,dw$; se $(x,y)\in A_z$, $x^2+y^2\le 4-z^2$

$\Leftrightarrow$ $(1+z^2)t^2 + (1+z^2)w^2 \le 4-z^2$

$\Leftrightarrow$ $t^2+w^2 \le \frac{4-z^2}{1+z^2}$. NE SEGUE $B_z = \left\{ t^2+w^2 \le \frac{4-z^2}{1+z^2} \right\}$ (R_z^2)

$$\circledast = \frac{1}{1+z^2}\cdot(1+z^2)\iint_B \frac{dt\,dw}{1+t^2+w^2}$$

$$= \iint_{B_2} \frac{dt\,dw}{1+t^2+w^2} = \text{(coord. polari)} = \int_0^{2\pi}\left(\int_0^{R_2} \frac{\rho\,d\rho}{1+\rho^2}\right)d\theta =$$

$$2\pi \int_0^{R_2} \frac{\rho}{1+\rho^2}\,d\rho = \pi \int_0^{R_2^2} \frac{ds}{1+s} = \qquad (\rho^2 = s)$$

$$\pi \ln(1+s)\Big|_0^{R_2^2} = \pi \ln\left(1+R_2^2\right) = \pi \ln\left(1+\left(\frac{4-z^2}{1+z^2}\right)\right)$$

ADESSO DOVREI CALCOLARE:

$$\pi \int_0^2 z \ln\left(1+\frac{4-z^2}{1+z^2}\right) dz = \qquad v = z^2 \qquad 2z\,dz = dv$$

$$= \frac{\pi}{2} \int_0^4 \ln\left(1+\frac{4-v}{1+v}\right) dv = \frac{\pi}{2} \int_0^4 \ln\left(\frac{5}{1+v}\right) dv =$$

$$\frac{\pi}{2} \int_0^4 \left(\ln(5) - \ln(1+v)\right) dv = 2\pi \ln 5 - \frac{\pi}{2} \int_0^4 \ln(1+v)\,dv$$

$$\int_0^4 \ln(1+v)\,dv = \left[(1+v)\ln(1+v)\right]_0^4 - \int_0^4 \frac{1+v}{1+v}\,dv \qquad \text{(per parti)}$$

$5\ln 5 - 4$. ALLA FINE TROVO

$$2\pi \ln 5 - \frac{\pi}{2} 5 \ln 5 + 2\pi = \boxed{-\frac{\pi}{2}\ln(5) + 2\pi} \qquad 2\pi - \frac{5}{2}\pi$$

UN ALTRO MODO (MIGLIORE) È DI USARE LE "COORDINATE SFERICHE".

CONSIDERO

$$x = \rho \cos\theta \sin\psi \qquad \text{per } \rho \geq 0$$
$$y = \rho \sin\theta \sin\psi \qquad 0 \leq \theta \leq 2\pi$$
$$z = \rho \cos\psi \qquad 0 \leq \psi \leq \pi$$

$(x,y,z) = \Psi(\rho,\theta,\psi)$ E MI SERVE LO JACOBIANO DI Ψ

$$J_\Psi(\rho,\theta,\psi) = \begin{pmatrix} \cos\theta \sin\psi & -\rho \sin\theta \sin\psi & \rho \cos\theta \cos\psi \\ \sin\theta \sin\psi & \rho \cos\theta \sin\psi & \rho \sin\theta \cos\psi \\ \cos\psi & 0 & -\rho \sin\psi \end{pmatrix}$$

$$\det J_\Psi(\rho,\theta,\psi) = \rho^2 \sin\psi \det \begin{pmatrix} \cos\theta\sin\psi & -\sin\theta & \cos\theta\cos\psi \\ \sin\theta\sin\psi & \cos\theta & \sin\theta\cos\psi \\ \cos\psi & 0 & -\sin\psi \end{pmatrix} =$$

$$\rho^2\sin\psi\left(\cos\psi\left(-\sin^2\theta\cos\psi - \cos^2\theta\cos\psi\right) - \sin\psi\left(\cos^2\theta\sin\psi + \sin^2\theta\sin\psi\right)\right)$$

$$= \rho^2\sin\psi\left(\cos^2\psi(-1) - \sin^2(\psi)\right) = -\rho^2\sin\psi$$

DUNQUE " $dx\,dy\,dz = \rho^2\sin(\psi)\,d\rho\,d\theta\,d\psi$ "

SE RIFACCIO L'INTEGRALE CON QUESTO CAMBIO DI VAR. TROVO

$$\iiint_B \frac{\rho\cos\psi}{1+\rho^2}\,\rho^2\sin\psi\,d\rho\,d\theta\,d\psi$$

con $B = \{\rho^2 \le 4,\ \rho\cos\psi \ge 0\} = \{0 \le \rho \le 2,\ 0 \le \psi \le \pi/2\}$

$$\to \underbrace{2\pi \int_0^{\pi/2} \sin\psi\cos\psi\,d\psi}_{①} \underbrace{\int_0^2 \frac{\rho^3}{1+\rho^2}\,d\rho}_{②} =$$

$$① = \pi\int_0^{\pi/2}\sin(2\psi)\,d\psi =$$

$$\pi\left[\frac{-\cos(2\psi)}{2}\right]_0^{\pi/2} = \frac{\pi}{2}(1+1) = \pi$$

$$② = \int_0^2 \rho\,\frac{\rho^2}{1+\rho^2}\,d\rho = \frac{1}{2}\int_0^4 \frac{s}{1+s}\,ds = \quad (s=\rho^2)$$

$$\frac{1}{2}\int_0^4\left(1 - \frac{1}{1+s}\right)ds = 2 - \frac{1}{2}\left[\ln(1+s)\right]_0^4 = 2 - \frac{1}{2}\ln(5)$$

IN DEFINITIVA $\pi\left(2 - \frac{1}{2}\ln(5)\right)$ che coincide con quanto trovato prima !!