Claudio Saccon (*)
Ingegneria Aerospaziale

Lezioni di Analisi Matematica 2 e Complementi

Lezione 22 14/11/17

(*) ricevimento il venerdì alle 10.30 al Dip. Matematica
email: claudio.sacconCHIOCCIOLAunipi.it
web: http://pagine.dm.unipi.it/csblog1/

ESERCIZIO (derivata della composizione)

Consideriamo $f : \mathbb{R}^3 \to \mathbb{R}^2$ e $g : \mathbb{R}^2 \to \mathbb{R}^2$ definite da

$$f(x,y,z) = \begin{pmatrix} 2x\, e^{yz} - y\, e^{xz} + 3z + 1 \\ z\, e^{xy} + 5x - 2y + 1 \end{pmatrix} \begin{matrix} \leftarrow f_1(x,y,z) \\ \leftarrow f_2(x,y,z) \end{matrix}$$

$$g(u,v) = \begin{pmatrix} u\, v^2 \\ u^2 v \end{pmatrix}$$

. Poniamo $h = g \circ f$ cioè $h(x,y,z) = g(f(x,y,z)) = g(f_1(x,y,z), f_2(x,y,z))$

$h : \mathbb{R}^3 \to \mathbb{R}^2$

DOMANDA 1 Calcolare $\frac{\partial h_2}{\partial y}(0,0,0) = ?!$

DOMANDA 2 Calcolare la matrice Jacobiana di h in $(0,0,0)$

Faccio direttamente la 2. . Noto che $f(0,0,0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Mi servono $J_f(0,0,0)$ e $J_g(1,1)$. ALLORA

$$J_f(x,y,z) = \begin{pmatrix} 2\, e^{yz} - yz\, e^{xz} & , & 2xz\, e^{yz} - e^{xz} & , & 2xy\, e^{yz} - xy\, e^{xz} + 3 \\ z\, y\, e^{xy} + 5 & , & z x\, e^{xy} - 2 & , & e^{xy} \end{pmatrix}$$

$$J_g(u,v) = \begin{pmatrix} v^2 & 2uv \\ 2uv & u^2 \end{pmatrix}$$

$$J_g(0,0,0) = \begin{pmatrix} 2 & -1 & 3 \\ 5 & -2 & 1 \end{pmatrix} \qquad J_g(1,1) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

TEORIA $\Rightarrow$

$$J_h(0,0,0) = J_g(1,1)\, J_h(0,0,0) =$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 & 3 \\ 5 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 2+10 & -1-4 & 3+2 \\ 4+5 & -2-2 & 6+1 \end{pmatrix} = \begin{pmatrix} 12 & -5 & 5 \\ 9 & -4 & 7 \end{pmatrix}$$

DUNQUE $\dfrac{\partial h_2}{\partial y}(0,0,0) = -4$

Si poteva usare formula:

$$\frac{\partial (g \circ f)_2}{\partial y}(0,0,0) = \frac{\partial g_2}{\partial u}(1,1)\cdot\frac{\partial f_1}{\partial y}(0,0,0) + \frac{\partial g_2}{\partial v}(1,1)\cdot\frac{\partial f_2}{\partial y}(0,0,0) = -4$$

$\frac{\partial g_2}{\partial u}(1,1) = 2$ $\quad \frac{\partial f_1}{\partial y}(0,0,0) = -1$ $\quad \frac{\partial g_2}{\partial v}(1,1) = 1$ $\quad \frac{\partial f_2}{\partial y}(0,0,0) = -2$

Esercizio (Taylor)

$$f(x,y,z) = 2\, e^{(x-y)^2}$$

(1) VOGLIO IL POLINOMIO DI ORDINE 3 in $P_0 = (1,1,1)$

(2) INOLTRE VOGLIO $\dfrac{\partial^3 f}{\partial x \partial y \partial z}(1,1,1)$

Conviene "traslare" il problema in $(0,0,0)$ scrivendo

$x = 1+\alpha \qquad y = 1+\beta \qquad z = 1+\gamma$

1 si semplifica

$$g(\alpha,\beta,\gamma) = f(1+\alpha, 1+\beta, 1+\gamma) = (1+\gamma)\, e^{(\alpha-\beta)^2}$$

CERCO LO SVILUPPO DI g in $(0,0,0) = O$. Uso lo sviluppo di Taylor di e^t (in una variabile)

$$e^t = 1 + t + O(t^2) \quad \Rightarrow$$

$$g(\alpha,\beta,\gamma) = (1+\gamma)\left[1 + (\alpha-\beta)^2 + \underbrace{O((\alpha-\beta)^4)}_{O(\|P\|^3)}\right] = (*)$$

RICORDO CHE $\quad |\gamma| = O(\|P\|)$, $|\beta| = O(\|P\|)$, $|\gamma| = O(\|P\|)$ $\Rightarrow$ $|\alpha-\beta| = O(\|P\|)$ $\qquad P = (\alpha,\beta,\gamma)$

$|\alpha-\beta|^4 = O(\|P\|^4)$ $\quad O((\alpha-\beta)^4) = O(\|P\|^4) = o(\|P\|^3)$ $\quad \|P-0\|^4$

$$(*) = 1+\gamma + (1+\gamma)(\alpha-\beta)^2 + \underbrace{o(\|P\|^3)(1+\gamma)}_{o(\|P\|^3)}$$

DUNQUE IL POLINOMIO IN 0 per g di ordine 3 è

$$1+\gamma+(1+\gamma)(\alpha^2-2\alpha\beta+\beta^2) =$$
$$1+\gamma+\alpha^2-2\alpha\beta+\beta^2+\gamma\alpha^2-2\alpha\beta\gamma+\gamma\beta^2$$

Se TORNO INDIETRO $(\alpha = x-1 \quad \beta = y-1 \quad \gamma = z-1)$

OTTENGO IL POLINOMIO di ordine 3 per f in $(1,1,1)$ cioè:

$$1+(z-1)+\underbrace{(x-1)^2}_{①}+\underbrace{(y-1)^2}_{②}-\underbrace{2(x-1)(y-1)}_{③}+\underbrace{(x-1)^2(z-1)}+(y-1)^2(z-1)+$$
$$-2(x-1)(y-1)(z-1)$$

(2) IN PARTICOLARE

$$-2 = \frac{1}{(1,1,1)!}\frac{\partial^3 f}{\partial x\partial y\partial z}(1,1,1) \Rightarrow \frac{\partial^3 f}{\partial x\partial y\partial z}(1,1,1) = -2$$

(1)

- SE VOLESSI $\frac{\partial^3 f}{\partial x^2\partial z}(1,1,1)$ DOVREI USARE IL MULTIINDICE $(2,0,1)$ e la formula

$$1 = \underbrace{\frac{1}{(2,0,1)!}}_{2}\frac{\partial^3 f}{\partial x^2\partial z}(1,1,1) \Rightarrow \frac{\partial^3 f}{\partial x^2\partial z}(1,1,1) = 2$$

- SI VEDE ANCHE $\frac{\partial^3 f}{\partial x^3}(P_0) = \frac{\partial^3 f}{\partial y^3}(P_0) = \frac{\partial^3 f}{\partial z^3}(P_0) = 0$

VEDIAMO ANCHE LE DERIVATE II

- $\frac{\partial^2 f(P_0)}{\partial x^2}\underbrace{\frac{1}{(2,0)!}}_{2} = 1 \Rightarrow \frac{\partial^2 f(P_0)}{\partial x^2} = 2$ ①

$\frac{\partial^2 f(P_0)}{\partial y^2}\frac{1}{(2,0)!} = 1 \Rightarrow \frac{\partial^2 f(P_0)}{\partial y^2} = 2$ ②

$\frac{\partial^2 f(P_0)}{\partial x\partial y}\frac{1}{(1,1)!} = -2 \Rightarrow \frac{\partial^2 f(P_0)}{\partial x\partial y} = -2$ ③

$$f(x,y,z) = z e^{(x-y)^2} = (z-1+1)\, e^{((x-1)+(1-y))^2} =$$

$$(z-1)\, e^{((x-1)+(1-y))^2} + e$$

$$x^2 = (1+x-1)^2 = 1 + \underbrace{2(x-1)}_{f'(1)} + \underbrace{\frac{1}{2}(x-1)^2}_{\frac{f''(1)}{2}}$$