Analisi Matematica II
Lezione 62
16 maggio 2016

Consideriamo

$$S := \{ z = 4x^2 + y^2 \,,\ x^2 + y^2 \le 1 \}$$

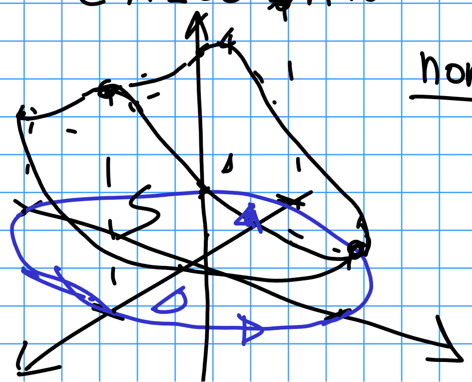
$$C := \{ 0 \le z \le 4x^2 + y^2 \,,\ x^2 + y^2 = 1 \}$$

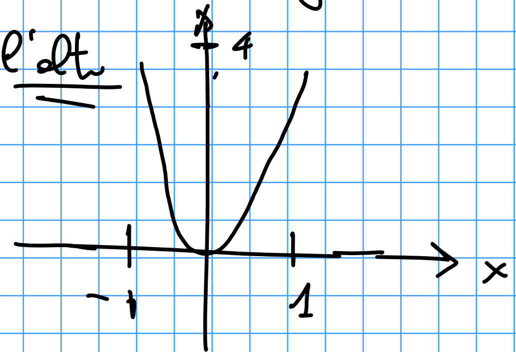
$$\Omega := \{ 0 \le z \le 4x^2 + y^2 \,,\ x^2 + y^2 \le 1 \}$$

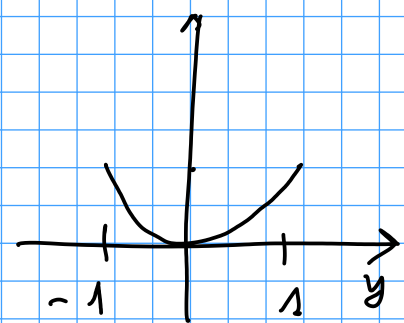
$$\vec{f}(x, y, z) := y(z + 3y^2)\,\vec{i} + x(z - 3x^2)\,\vec{j} + \frac{x^2 y^2}{1 + x^2 + y^2}\,\vec{k}$$

(A) CALCOLARE IL FLUSSO DI $\vec{f}$ attraverso S

normale verso l'alto







S è il grafico della funzione $z = g(x,y) := 4x^2 + y^2$

sul dominio $D = \{x^2 + y^2 \leq 1\}$. La posso parametrizzare

con $T(u,v) = u\vec{i} + v\vec{j} + g(u,v)\vec{k} \Rightarrow$

$$\vec{N}(u,u) = -\frac{\partial}{\partial u} g\,\vec{i} - \frac{\partial}{\partial v} g\,\vec{j} + \vec{k} =$$

$$- 8u\,\vec{i} - 2v\,\vec{j} + \vec{k}$$ ← NORMALE CONCORDE CON $\vec{k}$

Applicando la def. di flusso

$$\Phi(\vec{f}, S) = \iint_D \vec{f}(u, v, 4u^2+v^2) \cdot (-8u\vec{i} - 2v\vec{j} + \vec{k})\, du\, dv$$

$$= \iint_D \Big\{ v(\underbrace{4u^2+v^2}_{z} + 3v^2)(-8u) + u(\underbrace{4u^2+v^2}_{z} - 3u^2) \cdot (-2v)$$

$$+ \frac{u^2 v^2}{1+u^2+v^2} \cdot 1 \Big\}\, du\, dv =$$

$$= \iint_D \Big\{ -8uv\, 4(u^2+v^2) - 2uv(u^2+v^2) + \frac{u^2v^2}{1+u^2+v^2} \Big\}\, du\, dv$$

$$= \iint_D \underbrace{-34\, uv\,(u^2+v^2)\, du\, dv}_{=0} + \underbrace{\iint_D \frac{u^2 v^2}{1+u^2+v^2}\, du\, dv}_{\text{VEDI ALLA FINE}} = \text{(coord. polari)}$$

(per simmetria rispetto all'asse x o all'asse y)

$$\int_0^{2\pi} d\theta \int_0^1 \rho\, d\rho\, (-34)\, \rho\cos\theta\, \rho \sin\rho\, \rho^2 + \int_0^{2\pi} d\theta \int_0^1 \rho\, d\rho\, \frac{\rho^2\cos^2\theta\, \rho^2 \sin^2}{1+\rho^2}$$

$$= -34 \int_0^{2\pi} \cos\theta \sin\theta \int_0^1 \rho^5\, d\rho + \int_0^{2\pi} \cos^2\theta \sin^2\theta \int_0^1 \frac{\rho^5}{1+\rho^2}\, d\rho =$$

$$\underbrace{-17 \int_0^{2\pi} \sin(2\theta)\, d\rho}_{=0} \left[\frac{\rho^6}{6}\right]_0^1 + \frac{1}{4} \underbrace{\int_0^{2\pi} \sin^2(2\theta)\, d\theta}_{(A)} \underbrace{\int_0^1 \frac{\rho^5}{1+\rho^2}\, d\rho}_{(B)}$$

$$(A) = \frac{1}{4} \int_0^{2\pi} \frac{1-\cos(4\theta)}{2}\, d\theta = \frac{1}{4}\pi$$

$$(B) = \int_0^1 \frac{\rho^5+\rho^3-\rho^3-\rho+\rho}{1+\rho^2}\, d\rho = \int_0^1 \left(\rho^3-\rho+\frac{\rho}{1+\rho^2}\right) d\rho =$$

$$\left[\frac{\rho^4}{4} - \frac{\rho^2}{2} + \frac{1}{2}\ln(1+\rho^2)\right]_0^1 = \frac{1}{4} - \frac{1}{2} + \frac{\ln(2)}{2} = \frac{2\ln(2)-1}{4}$$

$$\Rightarrow \phi(\vec{f}, S) = \frac{\pi}{16}\left(2\ln(2) - 1\right) = \frac{\pi}{16}\left(\ln(4) - 1\right)$$

(B) Dire se esiste un potenziale vettore per $\vec{f}$ e calcolarne uno se possibile.

CONDIZIONE : $\operatorname{div} \vec{f} = 0$. Vediamo se è zero.

$$\underbrace{\frac{\partial}{\partial x} y(z+3y^2)}_{=0} + \underbrace{\frac{\partial}{\partial y} x(z-3x^2)}_{=0} + \underbrace{\frac{\partial}{\partial z} \frac{x^2+y^2}{1+x^2+y^2}}_{=0} = 0$$

TORNA : $\exists\ \vec{F} : \vec{\nabla} \otimes \vec{F} = \vec{f}$.

Per calcolare $\vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}$ suppongo $F_3 = 0$

$\to$ CONDIZIONI

$$\frac{\partial}{\partial z} F_1 = f_2 \ ; \quad \frac{\partial}{\partial z} F_2 = -f_1 \ ; \quad \frac{\partial}{\partial x} F_2 - \frac{\partial}{\partial y} F_1 = f_3$$

(1) $F_1(x,y,z) = \int x(z-3x^2)\,dz = \frac{xz^2}{2} - 3x^3 z + c(x,y)$

(2) $F_2(x,y,z) = -\int y(z+3y^2)\,dz = -y\frac{z^2}{2} - 3y^3 z + d(x,y)$

(3) IMPONGO L'ULTIMA RELAZIONE :

$$\frac{x^2y^2}{1+x^2+y^2} = \frac{\partial}{\partial x}\left(-\frac{yz^2}{2} - 3y^3z + d(x,y)\right) - \frac{\partial}{\partial y}\left(\frac{xz^2}{2} - 3x^3z + c(x,y)\right)$$

$$\Leftrightarrow \quad \frac{\partial}{\partial x} d(x,y) - \frac{\partial}{\partial y} c(x,y) = \frac{x^2y^2}{1+x^2+y^2}$$

CI SONO MOLTISSIMI MODI DI VERIFICARE L'ULTIMA

PER ESEMPIO $c = 0$ e

$$d(x,y) = \int \frac{x^2y^2}{1+x^2+y^2}\,dx = \frac{y^2}{1+y^2}\int \frac{x^2}{1+\frac{x^2}{1+y^2}}\,dx =$$

$$y^2\int \frac{\frac{x^2}{1+y^2}(+1-1)}{1+\frac{x^2}{1+y^2}}\,dx = y^2\int\left(1 - \frac{1}{1+\frac{x^2}{1+y^2}}\right)dx =$$

$$y^2\left(x - \sqrt{1+y^2}\,\arctan\left(\frac{x}{\sqrt{1+y^2}}\right)\right) + c$$

- ALLA FINE

$$\vec{F} = \left(\frac{xz^2}{2} - 3x^3z\right)\vec{i} + \left(\frac{yz^2}{2} - 3y^3z + xy^2 - \sqrt{1+y^2}\,\arctan\left(\frac{x}{\sqrt{1+y^2}}\right)\right)\vec{k}$$

$$(+\ \nabla G)$$

(C) PARAMETRIZZARE IL BORDO DI S, mediante una curva γ coerente con la normale

BASTA PRENDERE

$$\gamma(\theta) = \cos(\theta)\,\vec{i} + \sin(\theta)\,\vec{j} + g(\cos\theta, \sin\theta)\,\vec{K} =$$

$$(\Rightarrow \Gamma(\cos\theta, \sin\theta))$$

$$\cos(\theta)\,\vec{i} + \sin\theta\,\vec{j} + \left(4\cos^2\theta + \sin^2\theta\right)\vec{K} =$$

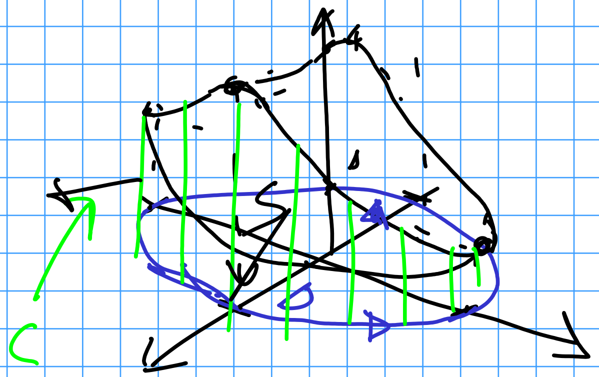
$$\cos(\theta)\,\vec{i} + \sin(\theta)\,\vec{j} + \left(4\,\frac{1+\cos(2\theta)}{2} + \frac{1-\cos(2\theta)}{2}\right)\vec{K} =$$

$$\cos(\theta)\,\vec{i} + \sin(\theta)\,\vec{j} + \left(\frac{5}{2} + \frac{3}{2}\cos(2\theta)\right)\vec{K}$$

(D) Calcolo $\int_\gamma \vec{F} \cdot d\vec{s}$ $\left(= \int_\gamma \vec{\nabla} \otimes \vec{f} \cdot d\vec{s} \right)$

SE USO STOKES: $\int_\gamma \vec{F} \cdot d\vec{s} = \Phi(\vec{f}, S) = \pi\,\frac{\ln(4)-1}{16}$

(E) Calcolare il flusso $\Phi(\vec{f}, C)$ NORMALE CHE PUNTA VERSO L'ESTERNO



$$C=\{0\le z\le 4x^2+y^2,\ x^2+y^2=1\}$$

$$\Omega=\{0\le z\le 4x^2+y^2,\ x^2+y^2=1\}$$

SI PUÒ FARE MEDIANTE LA DEF. DI FLUSSO ... .

LO VOGLIO FARE USANDO IL TEOREMA DELLA DIVERGENZA. (applicato a Ω)

NOTA $\partial\Omega = S \cup C \cup D$ $\Longrightarrow$ $\{x^2+y^2\le 1,\ z=0\}$

$$\iiint_\Omega \operatorname{div}(\vec{f})\,dx\,dy\,dz=0$$

$$\Rightarrow \overset{\shortparallel}{\Phi(\partial\Omega,\vec{f})} = \Phi(S,\vec{f})+\Phi(C,\vec{f})-\Phi(D,\vec{f})$$

se in D considero la normale $\vec{k}$

DUNQUE $\Phi(C,\vec{f}) = \Phi(D,\vec{f})-\Phi(S,\vec{f}) =$

$$= \Phi(D,\vec{f}) - \frac{\pi}{16}(\ln 4-1)$$

MANCA $\Phi(D, \vec{f}) =$

$$\iint_D f_3(u,v)\, du\, dv = \iint_D \frac{u^2 v^2}{1+u^2+v^2}\, du\, dv = \frac{\pi}{16}(\ln 4 - 1)$$

CALCOLO GIÀ FATTO

$$\Rightarrow \boxed{\phi(\vec{f}, C) = 0}$$

FINE

ESERCIZIO Dini

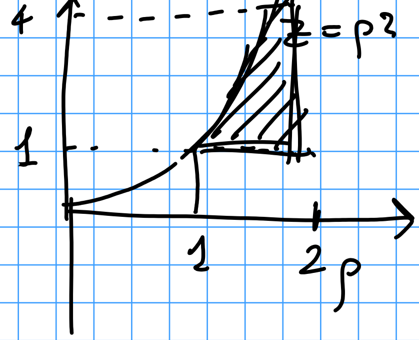
$\Omega := \{x^2+y^2 \le 4,\ 1 \le z \le x^2+y^2\}$

$S := \{1 \le x^2+y^2 \le 4,\ z = x^2+y^2\}$

$$\vec{f}(x,y,z) := \boxed{\frac{2y}{x^2+y^2}\vec{i} - \frac{3x}{x^2+y^2}\vec{j} - \vec{k}}$$

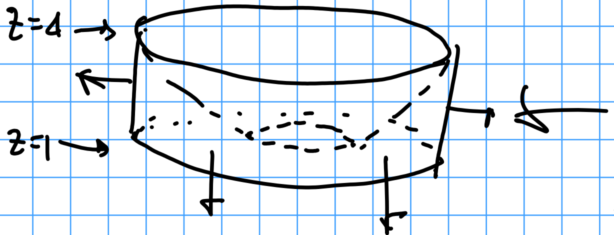
(A) DESCRIVERE $\partial\Omega$

SEZIONE



$\rho = \sqrt{x^2+y^2}$

$\Omega = \{\rho^2 \leq 4,\ 1 \leq z \leq \rho^2\}$



Ω $\quad \partial\Omega = S \cup C \cup A$ dove

$S = \{1 \leq x^2+y^2 \leq 4,\ z = x^2+y^2\}$

$C = \{x^2+y^2 = 4,\ 1 \leq z \leq 4\}$

$A = \{1 \leq x^2+y^2 \leq 4,\ z = 1\}$

(B) PARAMETRIZZARE S CON NORMALE CONCORDE CON $\vec{k}$

S è il grafico di $g(u,v) = u^2+v^2$ su

$D = \{(u,v) \in \mathbb{R}^2 : 1 \leq u^2+v^2 \leq 4\}$

$(A = \{(x,y,z) : (x,y) \in D,\ z = 1\})$

e quindi posso parametrizzare S mediante

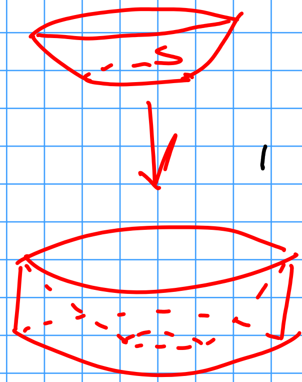
$\vec{r}(u,v) = u\,\vec{i} + v\,\vec{j} + (u^2+v^2)\,\vec{k} \qquad (u,v) \in D$

$\Rightarrow \vec{N}(u,v) = -2u\,\vec{i} - 2v\,\vec{j} + \vec{k}$ (concorde con $\vec{k}$!!)

(c) CALCOLARE IL FLUSSO $\Phi(\vec{f}, \partial\Omega)$

MEDIANTE LA DEFINIZIONE $\Phi(f,S)+\Phi(f,C)+\Phi(f,A)$

$$\Phi(\vec{f}, \underline{\underline{S}}) = \iint_D \vec{f}(u,v,u^2+v^2)\cdot \vec{N}(u,v)\,du\,dv =$$



$$\iint_D \left\{ \frac{2v}{u^2+v^2}\cdot(-2u) - \frac{3u}{u^2+v^2}(-2v) - 1\cdot 1\right\} du\,dv =$$

$$\iint_D \left(2\frac{uv}{u^2+v^2} - 1\right) du\,dv = -\iint_D du\,dv = -|D| =$$

fa zero per simmetria

(GIÀ VISTO SOPRA)

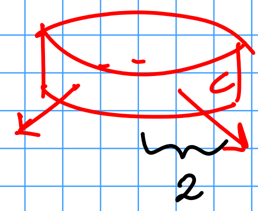
$$-\left(|B(0,2)| - |B(0,1)|\right) =$$

$$-\left(\pi\, 2^2 - \pi\cdot 1^2\right) = \boxed{-3\pi}$$

$$\Phi(\vec{f}, A) = \iint_D -f_3(x,y)\,dx\,dy$$ (su A devo mettere normale $-\vec{k}$)

$$= \iint_D 1\,dx\,dy = |D| = \pi\, 2^2 - \pi\, 1^2 = \boxed{3\pi}$$

$\Phi(\vec{f}, c)$. Descrivo c mediante $\Gamma(\theta, z) =$



$2\cos(\theta) i + 2\sin(\theta) j + z\vec{k} \qquad 0 \le \theta \le 2\pi \qquad 1 \le z \le 4$

$\frac{\partial \Gamma}{\partial \theta} = -2\sin\theta\,\vec{i} + 2\cos\theta\,\vec{j} \qquad \frac{\partial \Gamma}{\partial z} = \vec{k}$

$\vec{N} = \frac{\partial \Gamma}{\partial \theta} \otimes \frac{\partial \Gamma}{\partial z} = -\sin\theta\,(\underbrace{\vec{i} \otimes \vec{k}}_{-\vec{j}}) + \cos\theta\,(\underbrace{\vec{j} \otimes \vec{k}}_{\vec{i}}) = 2(\cos\theta\,\vec{i} + \sin\theta\,\vec{j})$

$(i \otimes k) \cdot j = -(k \otimes i) \cdot j = -(i \otimes j) \cdot \vec{k} = -1$

$\Rightarrow \Phi(\vec{f}, c) = \int_0^{2\pi} d\theta \int_1^4 dz \; \vec{f}(\cos\theta, \sin\theta, z) \cdot \vec{N}(\theta, z) =$

$\int_0^{2\pi} d\theta \int_1^4 dz \left(\frac{2\sin\theta}{\cancel{2}} \cdot \cancel{2}\cos\theta - \frac{3\cos\theta}{\cancel{2}} \cdot \cancel{2}\sin\theta + 0 \right) =$

$\int_0^{2\pi} (-\sin\theta\cos\theta)\, d\theta \int_1^4 dz = \boxed{0} \Rightarrow \Phi(\vec{f}, \partial\Omega) = -3\pi + 0 + 3\pi = 0$

$\int_0^{2\pi} \frac{\sin 2\theta}{2} = 0$

(D) CALCOLARE $\Phi(\vec{f}, \partial\Omega)$ usando il T. della divergenza

Calcolo $\operatorname{div}(\vec{f}) = \frac{\partial}{\partial x} \frac{2y}{x^2+y^2} + \frac{\partial}{\partial y} \frac{-3x}{x^2+2} + \frac{\partial}{\partial z}(-1) =$

$$2y\,\frac{-2x}{(x^2+y^2)^2} - 3x\,\frac{-2y}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

FACCIAMO $\iiint_\Omega \operatorname{div} \vec{f}\,dx\,dy\,dz =$

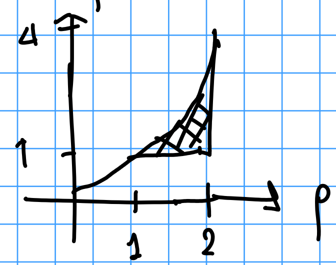
$$\iiint_\Omega \frac{2xy}{(x^2+y^2)^2}\,dx\,dy\,dz =$$

COORD CILINDRICHE

$z = z \quad x = \rho\cos\theta \quad y = \rho\sin\theta$

$$\int_0^{2\pi} d\theta \int_1^2 \rho\,d\rho \int_1^{\rho^2} \frac{2\rho^2\cos\theta\sin\theta}{\rho^4}\,dz =$$

$D \rightsquigarrow \{0 \le \theta \le 2\pi, \quad 1 \le z \le \rho^2 \le 4\}$



$$\underbrace{\int_0^{2\pi} \sin(2\theta)}_{=0} \int_1^2 \frac{d\rho}{\rho} \int_1^{\rho^2} dz = \boxed{0}$$

$\Rightarrow \quad \Phi(\vec{f}, \partial\Omega) = 0$