Analisi Matematica II
Lezione 58
4 maggio 2016

Flusso di $\vec{f}(x,y,z) = -y^2\,\vec{i} + x\,\vec{j} + z^3\,\vec{k}$

su (1) $S = \{x^2+y^2+z^2 = 1\}$

(2) $S^+ = \{x^2+y^2+z^2 = 1,\ z \geq 0\}$

(3) $S' = \{x^2+y^2+z^2 = 1,\ x \geq 0\}$

Nota che $\vec{\nabla} \cdot \vec{f} = \frac{\partial}{\partial x} y^2 - \frac{\partial}{\partial y} x + \frac{\partial}{\partial z} z^3 = 3z^2$ e $S = \partial B$

$B = \{x^2+y^2+z^2 \leq 1\}$

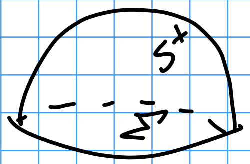
Dunque $\iint_S \vec{f} \cdot \hat{\nu}\, d\sigma = 3\iiint_B z^2\, dx\,dy\,dz =$

$3\int_{-1}^{1} z^2 \left(\iint_{x^2+y^2 \leq 1-z^2} dx\,dy \right) dz = 6\int_0^1 z^2\, \pi(1-z^2)\, dz =$

$\left(\iint_{x^2+y^2 \leq R^2} dx\,dy = \pi R^2 \right)$

$$6\pi\left[\frac{z^3}{3}-\frac{z^5}{5}\right]_0^1 = 6\pi\left(\frac{1}{3}-\frac{1}{5}\right) = 6\pi\frac{(5-3)}{15} = \frac{4}{5}\pi$$

(2)



Se $B^+ = \{x^2+y^2+z^2 \le 1,\ z \geq 0\} \Rightarrow$

$\partial B^+ = S^+ \cup \Sigma$ dove $\Sigma = \{x^2+y^2 \le 1,\ z=0\}$

$$\Rightarrow \iint_{S^+} \vec{f}\cdot\hat{\nu}\,d\sigma + \int_{\Sigma} f\cdot\hat{\nu}\,d\sigma = \iiint_{B^+} 3z^2\,dx\,dy\,dz = \frac{1}{2}\iiint_{B} 3z^2\,dx\,dy\,dz = \frac{2\pi}{5}$$

Dato che $\displaystyle\iint_{\Sigma} \vec{f}\cdot\vec{\nu} = \iint_{x^2+y^2\le 1} \vec{f}(x,y,0)\cdot(-\vec{k}) = \iint_{x^2+y^2\le 1} z^3\Big|_{z=0}\,dx\,dy = 0$

$$\Rightarrow \iint_{S^+} \vec{f}\cdot\hat{\nu}\,d\sigma = \frac{2\pi}{5}.$$

(3) Poniamo $\Sigma^1 = \{y^2+z^2 \le 1,\ x=0\}$ e calcoliamo

$$\iint_{\Sigma^1} \vec{f}\cdot\hat{\nu}\,d\sigma = \iint_{\{y^2+z^2\le 1\}} \vec{f}(0,y,z)\cdot(-\vec{i})\,dy\,dz =$$

$$\iint_{\{y^2+z^2\le 1\}} y^2\,dy\,dz = \int_0^{2\pi} d\theta \int_0^1 \rho^2 \sin^2\theta \cdot \rho\, d\rho =$$

$$\int_0^{2\pi} \sin^2\theta\, d\theta \left[\frac{\rho^4}{4}\right]_0^1 = \frac{1}{4}\int_0^{2\pi} \frac{1-\cos(2\theta)}{2}\, d\theta =$$

$$\frac{1}{4}\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{2\pi} = \frac{1}{4}\cdot\frac{2\pi}{2} = \frac{\pi}{4}$$

QUINDI $\displaystyle \iint_{S'} \vec{f}\cdot\hat{\nu}\, d\sigma + \iint_{\Sigma'} \vec{f}\cdot\hat{\nu}\, d\sigma = \iiint_{B'} 3z^2\,dx\,dy\,dz =$ (per simmetria)

$(B' = \{x^2+y^2+z^2 \le 1,\ x \ge 0\})$ $\displaystyle \frac{1}{2}\iiint_{B} 3z^2\,dx\,dy\,dz = \frac{2}{5}\pi$

da cui $\displaystyle \iint_{S'} \vec{f}\,\vec{\nu}\, d\sigma = \frac{2}{5}\pi - \frac{\pi}{4} = \frac{3}{20}\pi$

PROVIAMO – per completezza – a calcolare $\displaystyle \iint_{S} \vec{f}\cdot\hat{\nu}\, d\sigma$ mediante la definizione. Usiamo la parametrizzazione

$$\Gamma(\psi,\theta) = \cos(\theta)\sin(\psi)\vec{i} + \sin(\theta)\sin(\psi)\vec{j} + \cos(\psi)\vec{k} \qquad \begin{array}{l} 0\le\theta\le 2\pi \\ 0\le\psi\le\pi \end{array}$$

(che non è bigettiva, ma lo è se si escludono $\theta = 0/2\pi$ $\psi = 0/\pi$)

$$\frac{\partial\Gamma}{\partial\psi} = \cos(\theta)\cos(\psi)\vec{i} + \sin(\theta)\cos(\psi)\vec{j} - \sin(\psi)\vec{k}$$

$$\frac{\partial\Gamma}{\partial\theta} = -\sin(\theta)\sin(\psi)\vec{i} + \cos(\theta)\sin(\psi)\vec{j}$$

$$\frac{\partial\Gamma}{\partial\psi} \otimes \frac{\partial\Gamma}{\partial\theta} = \det\begin{bmatrix} \vec{i} & \cos(\theta)\cos(\psi) & -\sin(\theta)\sin(\psi) \\ \vec{j} & \sin(\theta)\cos(\psi) & \cos(\theta)\sin(\psi) \\ \vec{k} & -\sin(\psi) & 0 \end{bmatrix} =$$

(formalmente ..)

$$\cos(\theta)\sin^2(\psi)\vec{i} + \sin(\theta)\sin^2(\psi)\vec{j} + \sin(\psi)\cos(\psi)\vec{k}$$

DUNQUE: (se $R = \{0\le\psi\le\pi,\ 0\le\theta\le 2\pi\}$)

$$\iint_S \vec{f}\cdot\hat{\nu}\, d\sigma =$$

$$\iint_R -\left(\sin(\theta)\sin(\psi)\right)^2 \cos(\theta)\sin^2(\psi)\, d\theta\, d\psi +$$

$$+ \iint_R \cos(\theta)\sin(\psi)\sin(\theta)\sin^2\psi\, d\theta\, d\psi +$$

$$+\iint_R \left(\cos(\psi)\right)^3 \sin(\psi)\cos(\psi)\, d\theta\, d\psi =$$

$$-\int_0^{2\pi} \sin^2(\theta)\cos(\theta)\, d\theta \int_0^{\pi} \sin^4(\psi)\, d\psi +$$

$$\int_0^{2\pi} \sin(\theta)\cos(\theta)\, d\theta \int_0^{\pi} \sin^3(\psi)\, d\psi \quad + 2\pi \int_0^{\pi} \cos^4(\psi)\sin(\psi)\, d\psi = (1)+(2)+(3)$$

MA $(1) = 0$ dato che $\displaystyle\int_0^{2\pi} \sin^2(\theta)\cos(\theta)\, d\theta \underset{(s=\sin(\theta))}{=} \int_{\sin(0)}^{\sin(2\pi)} s^2\, ds = \int_0^0 s^2\, ds = 0$

$(2) = 0$ dato che $\displaystyle\int_0^{2\pi} \sin(\theta)\cos(\theta)\, d\theta \underset{(s=\sin(\theta))}{=} \int_{\sin(0)}^{\sin(2\pi)} s\, ds = \int_0^0 s\, ds = 0$

$$(3) = 2\pi \int_0^{\pi} \cos^4(\theta)\sin(\theta)\, d\theta \underset{(s=\cos(\theta))}{=} 2\pi \int_{\cos(0)}^{\cos(\pi)} s^4(-ds) =$$

$$2\pi \int_{-1}^{1} s^4\, ds = 4\pi \int_0^1 s^4\, ds = 4\pi \left[\frac{s^5}{5}\right]_0^1 = \frac{4\pi}{5} \quad !!!$$