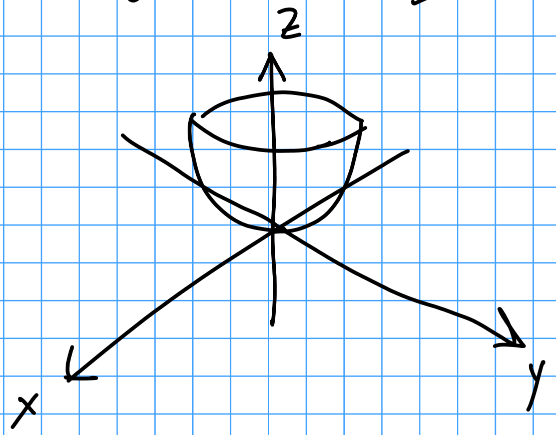
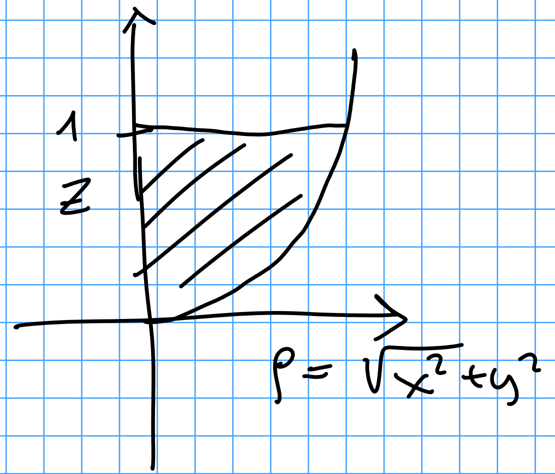
Analisi Matematica II
Lezione 30
7 dicembre 2015

$$\iiint_A z\, e^{x^2+y^2}\, dx\, dy\, dz \qquad \text{dove:}$$

$$A = \{ x^2+y^2 \le z \le 1 \} \subset \mathbb{R}^3$$





PASSAGGIO A COORDINATE CILINDRICHE:

$$\begin{cases} x = \rho\cos\vartheta \\ y = \rho\sin\vartheta \\ z = z \end{cases} \Rightarrow J_\phi(\rho,\vartheta,z) = \rho\, d\rho\, d\vartheta\, dz$$

PASSO ALL'INTEGRALE

$$\iiint_{A_1} z\, e^{\rho^2} \rho \, d\rho \, d\theta \, dz =$$

$\rho^2 \le z$

$$A_1 = \{ (\rho, \theta, z) : \quad 0 \le \theta \le 2\pi \quad 0 \le z \le 1 \quad 0 \le \rho \le \sqrt{z} \}$$

$$\int_0^{2\pi} d\theta \int_0^1 z\, dz \int_0^{\sqrt{z}} \rho\, e^{\rho^2} d\rho = \cancel{2}\pi \int_0^1 z\, dz \left[\frac{e^{\rho^2}}{\cancel{2}} \right]_0^{\sqrt{z}} =$$

$\sigma = \rho^2$

$d\sigma = 2\rho\, d\rho \Rightarrow \frac{1}{2}\int_0^z e^{\sigma} d\sigma$ //

$$\pi \left[\frac{z^2}{2} \right]_0^1 = \frac{\pi}{2}$$

$$\pi \int_0^1 z \left(e^z - 1 \right) dz = \pi \int_0^1 z e^z \, dz - \pi \int_0^1 z\, dz$$

$$\pi \left[z e^z \right]_0^1 - \pi \int_0^1 e^z \, dz - \frac{\pi}{2} = \pi (e - 0) - \pi \left[e^z \right]_0^1 - \frac{\pi}{2} =$$

$$\pi e - \pi e + \pi - \frac{\pi}{2} = \frac{\pi}{2}$$

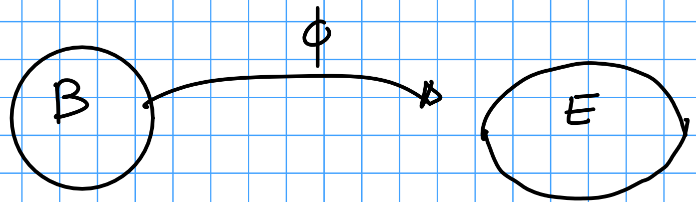
ESERCIZIO VOLUME DI $E = \{ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1 \}$

SCRIVO $x = a u \qquad y = b v \qquad z = c w \qquad$ ~ $\quad 0 \le u, v, w \le 1$

in astratto definisco $\phi(u,v,w) = (au, bv, cw)$

e in questo modo $E = \phi(B)$ dove

$B = \{u^2+v^2+w^2 \le 1\}$



$$J_\phi = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

$\det J_\phi = abc$

Applicando il cambio di variabile $\Longrightarrow$

$$\iiint_B \det J_\phi(u,v,w)\, du\, dv\, dw = \iiint_E dx\, dy\, dz = |E|$$

Dunque $|E| = abc\,|B| = 4\dfrac{abc}{3}$ #

$$\iiint_E (x^2+y^2)\, dx\, dy\, dz$$

$$E = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1 \right\}$$

I° cambio di variabile $x = au \quad y = bv \quad z = cw$

$$\to abc \iiint_B a^2u^2 + b^2v^2 \; du\, dv\, dw \;, \quad B = \{u^2+v^2+w^2 \le 1\}$$

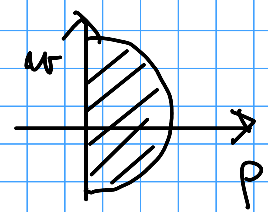
II° cambio di var. coordinate cilindriche

$$u = \rho \cos\theta \qquad v = \rho \sin\theta \qquad w = w$$

$$\iiint_{B_1} \left(a^2\rho^2\cos^2\theta + b^2\rho^2\sin^2\theta\right)\rho \, d\rho \, d\theta \, dw =$$

$$B_1 = \{0 \leq \theta \leq 2\pi, \quad \rho^2 + w^2 \leq 1\}$$

$$\int_0^{2\pi}\left(a^2\cos^2\theta + b^2\sin^2\theta\right)d\theta \iint_{\{\rho^2+w^2\leq 1, \, \rho\geq 0\}} \rho^3 \, d\rho \, dw =$$



$$\underbrace{\int_0^{2\pi}\left(a^2\cos^2\theta + b^2\sin^2\theta\right)d\theta}_{1} \underbrace{\int_{-1}^{1}\left(\int_0^{\sqrt{1-w^2}} \rho^3 \, d\rho\right)dw}_{2} = \star$$

$$1 = \int_0^{2\pi} \frac{a^2}{2}\left(1+\cos 2\theta\right) + \frac{b^2}{2}\left(1-\cos 2\theta\right) d\theta =$$

$$\underbrace{\int_0^{2\pi}\left(\frac{a^2}{2}-\frac{b^2}{2}\right)\cos(2\theta)\,d\theta}_{=0} + \left(\frac{a^2}{2}+\frac{b^2}{2}\right)2\pi = (a^2+b^2)\pi$$

$$(2) = \int_{-1}^{1}\left[\frac{\rho^4}{4}\right]_0^{\sqrt{1-w^2}} dw = \int_{-1}^{1}\frac{(1-w^2)^2}{4}\,dw =$$

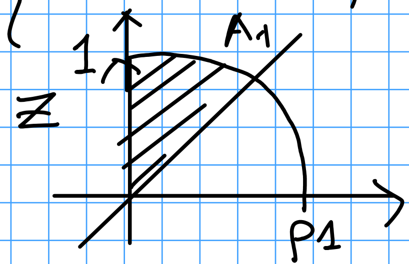
$$\frac{1}{2}\int_0^1 \left(1-2w^2+w^4\right)dw = \frac{1}{2}\left[w-\frac{2}{3}w^3+\frac{w^5}{5}\right]_0^1 =$$

$$\frac{1}{2}\left(1-\frac{2}{3}+\frac{1}{5}\right) = \frac{1}{2}\,\frac{15-10+3}{15} = \frac{1}{2}\,\frac{8}{15} = \frac{4}{15}$$

$$\Rightarrow (\star) = \frac{4\pi}{15}\left(a^2+b^2\right)$$

$$\iiint_A \frac{dx\,dy\,dz}{\sqrt{x^2+y^2}} \quad \text{dove} \quad A=\left\{x^2+y^2+z^2\le 1,\ x^2+y^2\le z^2,\ z\ge 0\right\}$$

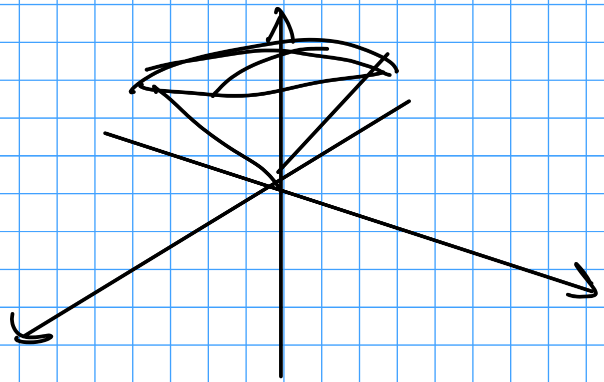
se chiamo $\rho=\sqrt{x^2+y^2} \Rightarrow$



$\rho^2+z^2\le 1$

$z\ge\rho$

$A =$ INTERSEZIONE TRA CONO $z \geq \sqrt{x^2+y^2}$
e LA PALLA $\{x^2+y^2+z^2 \leq 1\}$ (e il semipiano $z \geq 0$)



CONVIENE PASSAGGIO A COORD. CILINDRICHE

$x = \rho\cos\theta \quad y = \rho\sin\theta \quad z = z$

$$\longrightarrow \int_0^{2\pi} d\theta \iint_{A_1} \frac{1}{\rho}\, \rho \, d\rho\, dz = 2\pi |A_1|$$

$$A_1 = \{\rho \leq z \leq \sqrt{1-\rho^2}\} = \{0 \leq \rho \leq 1,\ \rho \leq z \leq \sqrt{1-\rho^2}\}$$

$$|A_1| = \int_0^{\sqrt{2}/2} d\rho \int_\rho^{\sqrt{1-\rho^2}} dz = \int_0^{\sqrt{2}/2} \left(\sqrt{1-\rho^2}-\rho\right) d\rho =$$

(deve venire $\frac{\pi}{8}$ ⇐ $\frac{1}{8}$ del cerchio unitario)

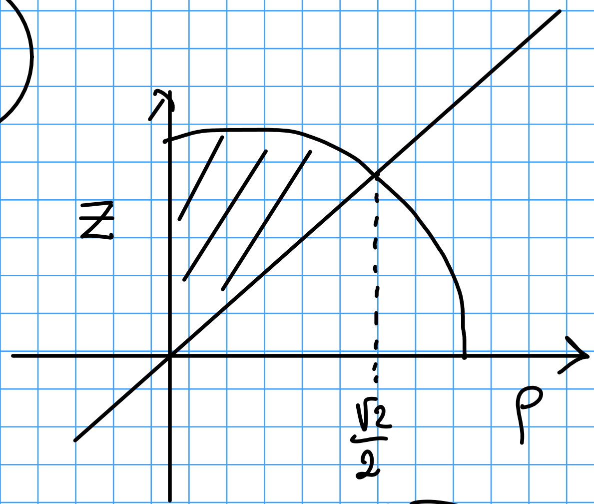
$$\int_0^{\sqrt{2}/2} \sqrt{1-\rho^2}\, d\rho - \left[\frac{\rho^2}{2}\right]_0^{\sqrt{2}/2} =$$

$\rho = \sin t \quad d\rho = \cos t\, dt$

$$\int_0^{\pi/4} \cos^2 t\, dt - 1/4 =$$

$$\int_0^{\pi/4} \frac{1+\cos(2t)}{2}\, dt - \frac{1}{4} = \frac{\pi}{8} + \frac{1}{2}\left[\frac{\sin(2t)}{2}\right]_0^{\frac{\pi}{4}} - \frac{1}{4} =$$

$$\frac{\pi}{8} + \frac{1}{4} - \frac{1}{4} = \frac{\pi}{8}$$

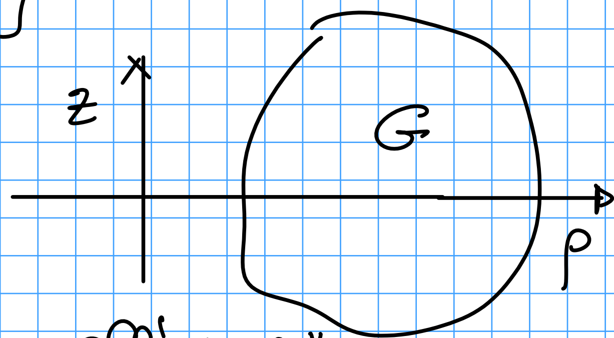


$\Rightarrow$ INTEGRALE $= \frac{\pi}{8} \cdot 2\pi = \frac{\pi^2}{4}$

SOLIDI DI ROTAZIONE: Supponiamo che:

$$\Omega = \{ (x,y,z) : (\sqrt{x^2+y^2}, z) \in G \}$$

dove $G \subset \{ (\rho, z) : \rho \geq 0 \}$



Ω "si ottiene da G, ruotando attorno all'asse z"

e che $g: G \to \mathbb{R}$. Allora

se $f(x,y,z) = g\left(\sqrt{x^2+y^2}, z\right)$ (f è "RADIALE") $\Rightarrow$

$$\iiint_{\Omega} f(x,y,z)\,dx\,dy\,dz \quad = \quad \text{coordinate cilindriche}$$
$(x = \rho\cos\theta \quad y = \rho\sin\theta \quad z = z)$

$$\iiint_{\Omega_1} g(\rho, z)\,\rho\,d\rho\,d\theta\,dz \quad =$$

$$\Omega_1 : \{0 \le \theta \le 2\pi, \ (\rho, z) \in G\}$$

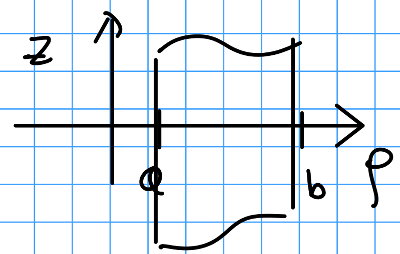
$$2\pi \iint_G g(\rho, z)\,\rho\,d\rho\,dz$$

IN PARTICOLARE $\quad |\Omega| = 2\pi \iint_G \rho\,d\rho\,dz$

- Se poi $G = \{a \le \rho \le b, \ \varphi_1(\rho) \le z \le \varphi_2(\rho)\} \Rightarrow$

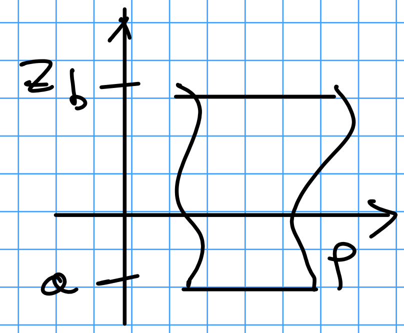
$$|\Omega| = 2\pi \int_0^b \rho \int_{\varphi_1(\rho)}^{\varphi_2(\rho)} dz =$$

$$2\pi \int_0^b \rho\,(\varphi_2(\rho) - \varphi_2(\rho))\,d\rho$$



- Se invece $G = \{ a \leq z \leq b, \ \underset{\geq 0}{\psi_1(z)} \leq \rho \leq \psi_2(z) \} \Rightarrow$

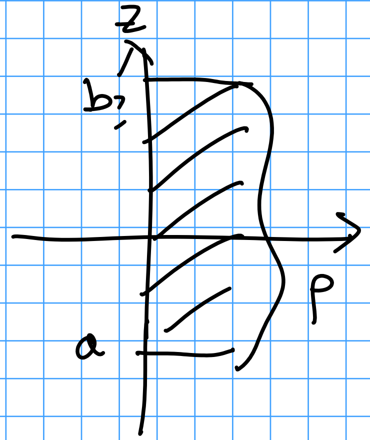
$$|\Omega| = 2\pi \int_a^b dz \int_{\psi_1(z)}^{\psi_2(z)} \rho \, d\rho =$$



$$2\pi \int_a^b dz \left[\frac{\rho^2}{2} \right]_{\psi_1(z)}^{\psi_2(z)} =$$

$$\pi \int_a^b \left(\psi_2(z)^2 - \psi_1(z)^2 \right) dz$$

CASO $\psi_1 = 0$, $\psi_2 = \psi$



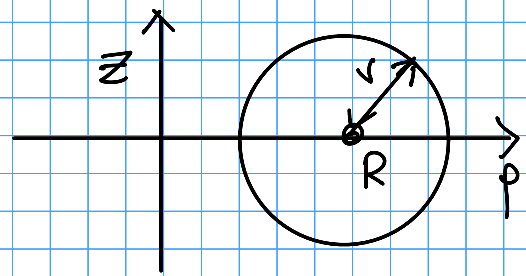
$$\Rightarrow \pi \int_a^b \psi^2(z) \, dz$$

$$G = \{ (\rho, z) : \ 0 \leq \rho \leq \psi(z) \}$$

e (più in generale)

$$\iiint_\Omega f(x,y,z) \, dx\,dy\,dz = \int_a^b dz \int_{\psi_1(z)}^{\psi_2(z)} \rho \, g(\rho, z) \, d\rho$$

USIAMO LA FORMULA PER CALCOLARE IL VOLUME DEL TORO $G =$ CERCHIO DI RAGGIO r e centro $(R, 0)$ con $R > r$



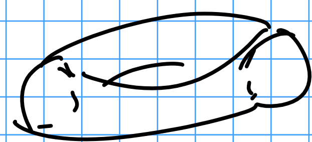
$$G = \left\{ -r \leq z \leq r \;,\; \underbrace{R - \sqrt{r^2 - z^2}}_{\psi_1} \leq \rho \leq \underbrace{R + \sqrt{r^2 - z^2}}_{\psi_2} \right\}$$

$$\mathrm{dist}\big((\rho, z), (R, 0) \big) \leq r \iff$$

$$(\rho - R)^2 + z^2 \leq r^2 \iff$$

$$(\rho - R)^2 \leq r^2 - z \iff$$

$$-\sqrt{r^2 - z^2} \leq \rho - R \leq \sqrt{r^2 - z^2}$$



DUNQUE, se T si ottiene da G ruotando

$$|T| = \left(\pi \int_{-r}^{r} \left(\psi_2^2(z) - \psi_1^2(z) \right) dz \right) dz$$

$$2\pi\int_0^r \left[\left(\underbrace{R+\sqrt{r^2-z^2}}_{\beta}\right)^2 - \left(\underbrace{R-\sqrt{r^2-z^2}}_{\alpha}\right)^2\right] dz =$$

$$2\pi\int_0^r \underbrace{2R}_{\beta+\alpha}\ \underbrace{2\sqrt{r^2-z^2}}_{\beta-\alpha}\, dz = 8\pi R r \int_0^r \sqrt{1-\left(\frac{z}{r}\right)^2}\, dz$$

$\frac{z}{r} = w$ $\quad dz = r\,dw \quad$ $= 8\pi R r^2 \int_0^1 \sqrt{1-w^2}\, dw = A$

$\int_0^1 \sqrt{1-w^2}\, dw \qquad w = \sin t \qquad dw = \cos t\, dt$

$$\int_0^{\pi/2} \cos^2 t\, dt = \int_0^{\pi/2} \frac{\cos 2t + 1}{2}\, dt = \frac{\pi}{4}$$

$$|T| = 2\pi^2 R r^2 = (2\pi R)\,\pi r^2$$

lunghezza dell' "anima" del toro

Volume della sezione

ESERCIZIO DI PRIMA

$$\iiint_{\Omega} \frac{dx\,dy\,dz}{\sqrt{x^2+y^2}} \qquad \Omega = \left\{ \begin{array}{l} x^2+y^2+z^2 \le 1 \\ x^2+y^2 \le z^2, \quad z \ge 0 \end{array} \right\}$$

Ω è ottenuto da G ruotando attorno all'asse z dove

$$G = \{ (\rho, z) : \rho^2 + z^2 \le 1, \quad \rho \le z \}$$

$$f(x,y,z) = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{\rho} \quad (= g(\rho, z))$$ DUNQUE

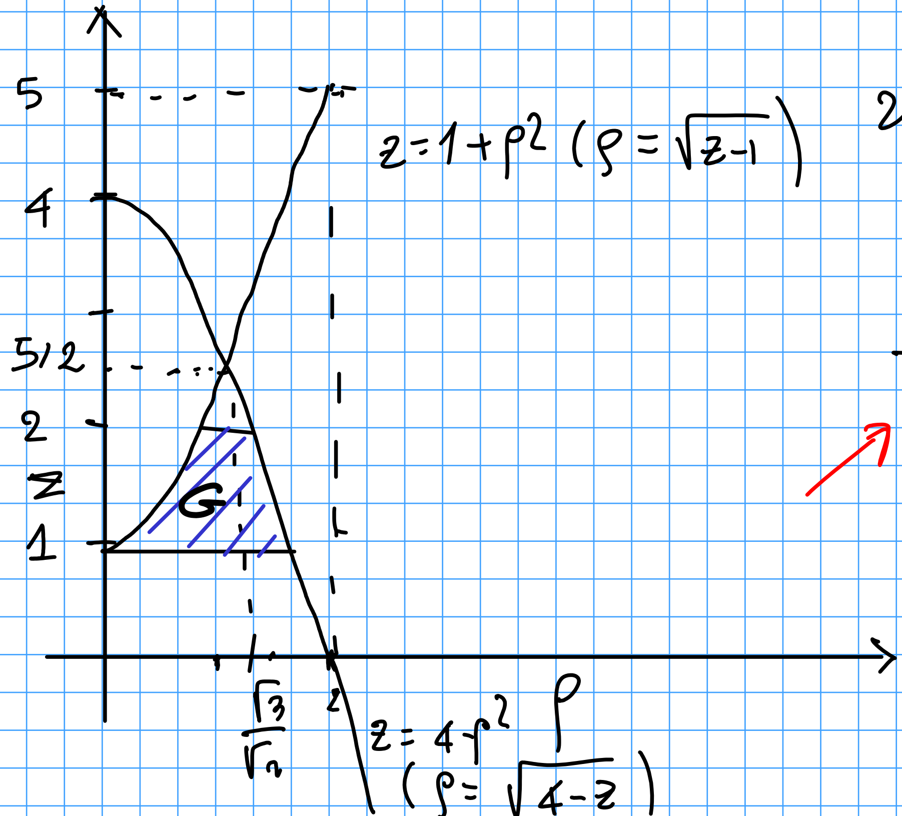
PER LE FORMULE VISTE

$$\text{INTEGRALE} = 2\pi \iint_G \rho \, \frac{1}{\rho} \, d\rho \, dz = 2\pi \, |G|$$

$$= \text{(calcoli già visti)} = 2\pi \, \frac{\pi}{8} = \frac{\pi^2}{4}$$

$$G = \{ (\rho, z) : \sqrt{z-1} \le \rho \le \sqrt{4-z}, \quad 1 \le z \le 2 \}$$

VOGLIO IL VOLUME DI Ω ottenuto ruotando G attorno all'asse z.



$$2\pi \int_1^2 dz \int_{\sqrt{z-1}}^{\sqrt{4-z}} \rho \, d\rho = \frac{2\pi}{2} \int_1^2 dz \left(\underbrace{4-z}_{\varphi_2} \underbrace{-z+1}_{\varphi_1} \right) =$$

$$\pi \int_1^2 (5-2z)\, dz = \pi \left(\left[5z - z^2 \right]_1^2 \right) =$$

$$\pi (5 - 4 + 1) = 2\pi$$

$$G = \{ a \le z \le b,\ \varphi_1(z) \le \rho \le \varphi_2(z) \}$$

$$\Big(|\Omega| = 2\pi \iint_G \rho \, d\rho \, dz = 2\pi \int_a^b dz \int_{\varphi_1(z)}^{\varphi_2(z)} \rho \, d\rho =$$

$$2\pi \int_a^b \left[\frac{\rho^2}{2} \right]_{\varphi_1(z)}^{\varphi_2(z)} dz = \pi \int_a^b \left(\varphi_2(z)^2 - \varphi_1(z)^2 \right) dz \Big)$$