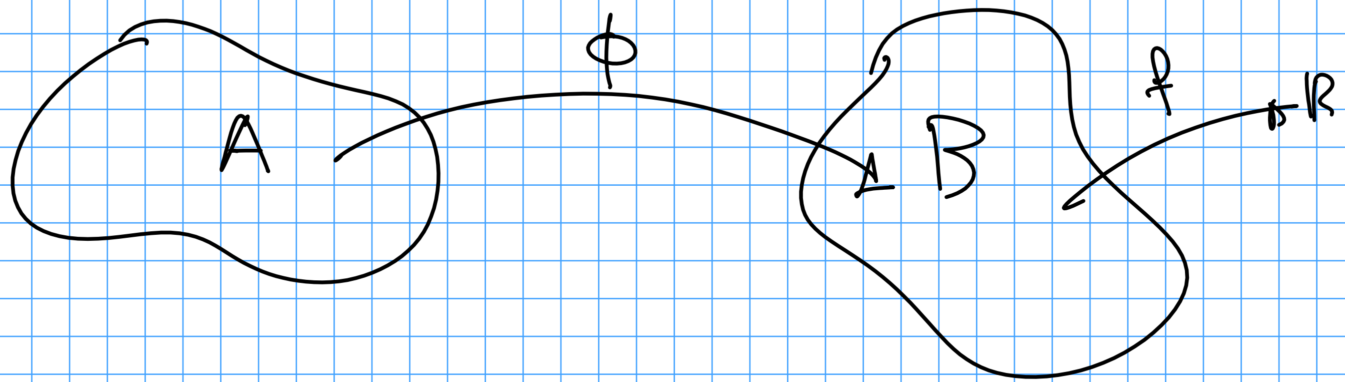
Analisi Matematica II
Lezione 29
2 dicembre 2015

FORMULA DI CAMBIO NEGLI INTEGRALI MULTIPLI

TEOREMA A e B due domini regolari chiusi e limitati in $\mathbb{R}^N$ e $\phi : A \to B$ di classe C^1 e bigettivo



Sia inoltre $f : B \to \mathbb{R}$ integrabile su B

ALLORA $f \circ \phi$ è integrabile su A

Inoltre

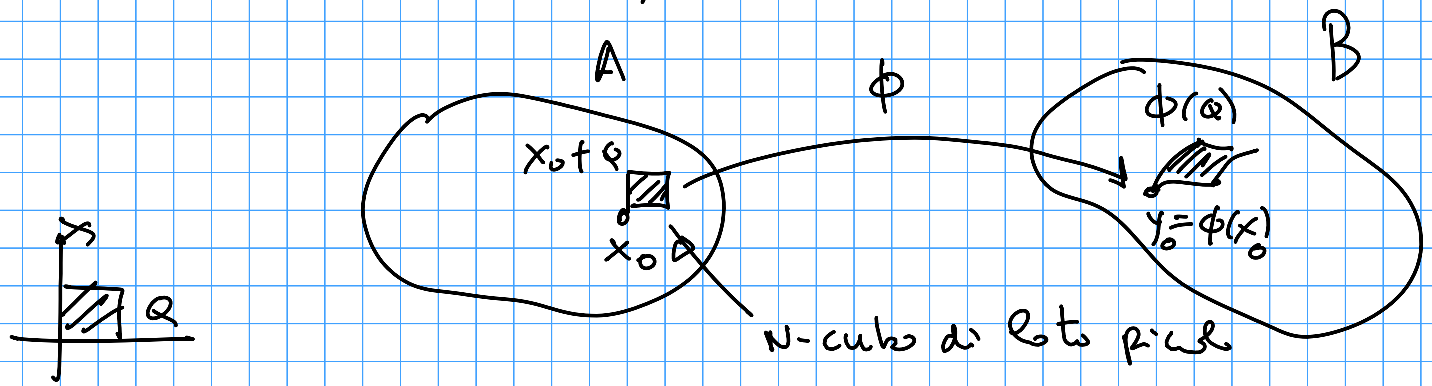
$$\int_B f(x)\,dx = \int_A f(\phi(y))\,|\det J_\phi(y)|\,dy$$

dove J_ϕ denota la matrice Jacobiana di ϕ $(N\times N)$

NO DIM.

In particolare, se $f(x)=1$, si ha

$$|B| = \int_A |\det J_\phi(y)|\,dy$$

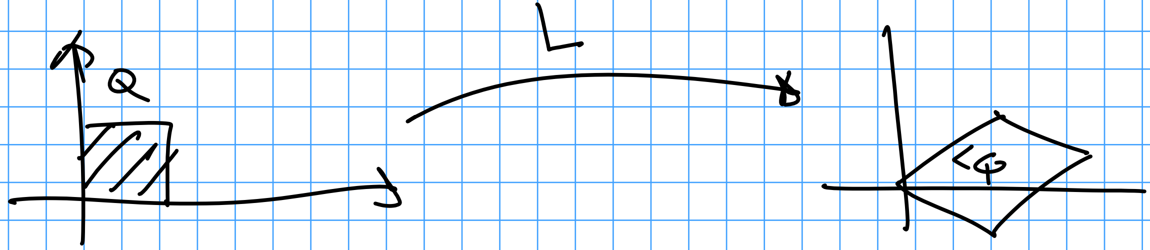


Q = cubetto di lato piccolo con vertice zero

se $x \simeq x_0$ $\quad \phi(x) \simeq \underbrace{\phi(x_0)}_{y_0} + \underbrace{J_\phi(x_0)(x-x_0)}_{\text{LINEARE}}$

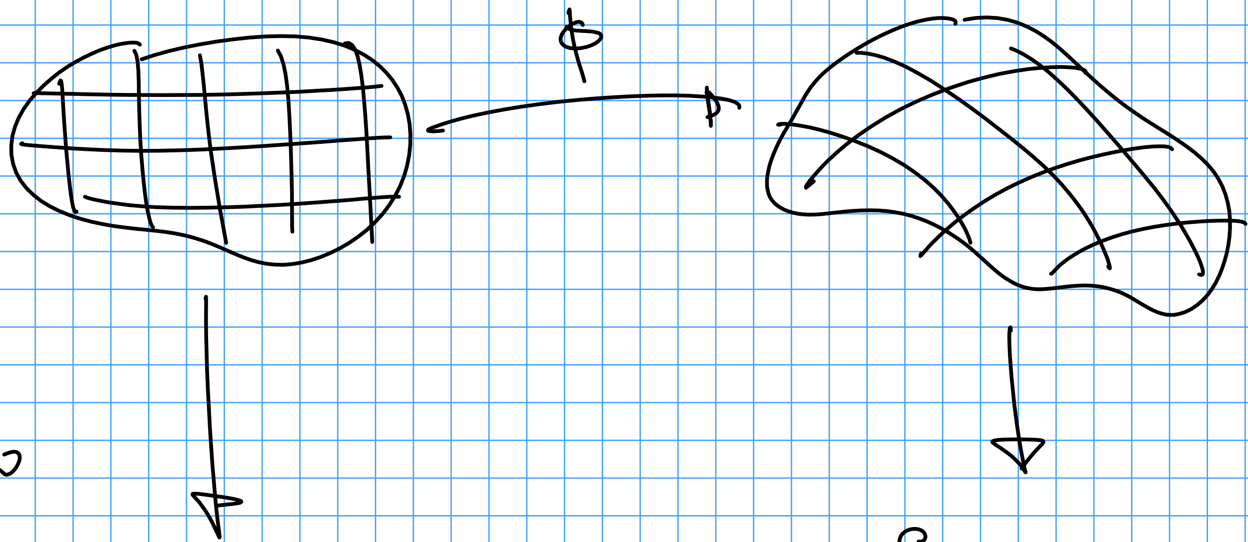
quindi $\phi(x_0+Q) \simeq y_0 + J_\phi(Q)$

$\Rightarrow \quad |\phi(x_0+Q)| \simeq |\det J_\phi|\,|Q|$ (PROPRIETÀ DELLE APPL. LINEARI)



$$|LQ| = |Q|\,|\det L|$$

Se lo faccio per ogni cubetto



Se $|Q| \to 0$

$$|A| \qquad\qquad \int |J_\phi|\, dy$$

IN PARTICOLARE, se $L : \mathbb{R}^N \to \mathbb{R}^N$ è lineare

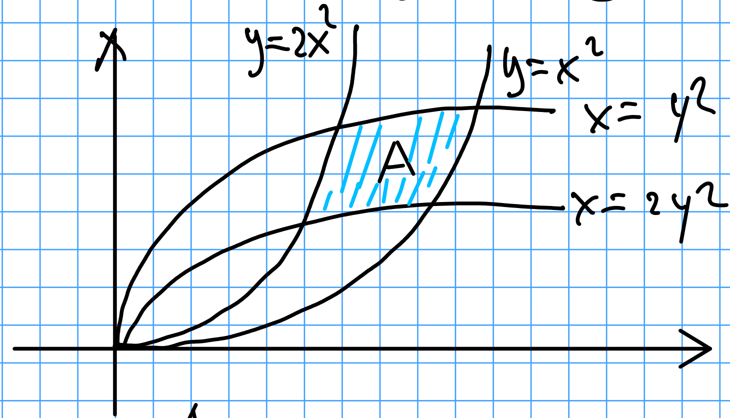
e $\det L \neq 0$

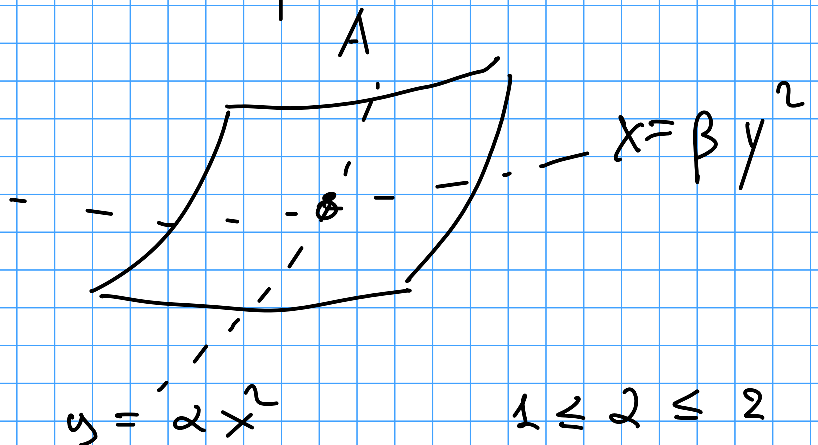
$B = LA$

$$\int_A f(x)\,dx = |\det L| \int_B f(Ly)\, dy$$

ESEMPI (es. 8 pag. 305 ADAMS)

Calcolo $|A|$ dove $A = \{ x^2 \le y \le 2x^2,\ y^2 \le x \le 2y^2 \}$





$1 \le \beta \le 2$

$1 \le \alpha \le 2$

"Voglio usare (α, β) come coordinate".

Al punto (x, y) associo (α, β) in modo che

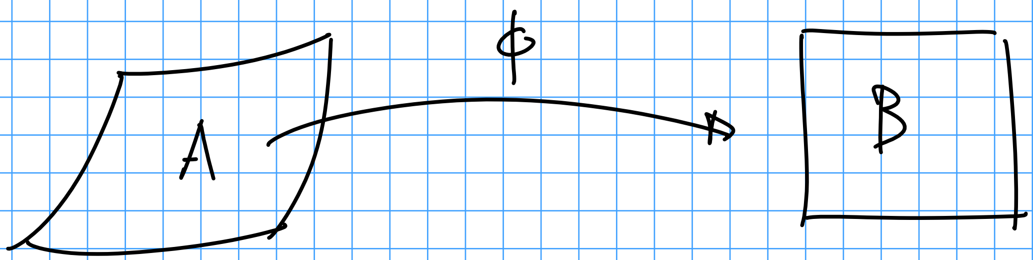
$\alpha = \dfrac{y}{x^2}$ $\quad \beta = \dfrac{x}{y^2}$ cioè $(\alpha,\beta) = \phi(x, y)$

dove $\phi(x,y) = \left(\frac{y}{x^2}, \frac{x}{y^2}\right)$ con $(x,y) \in A$

Calcolo $J\phi(x,y) = \begin{bmatrix} -\frac{2y}{x^3} & \frac{1}{x^2} \\ \frac{1}{y^2} & -\frac{2x}{y^3} \end{bmatrix} \Rightarrow \det J\phi(x,y) =$

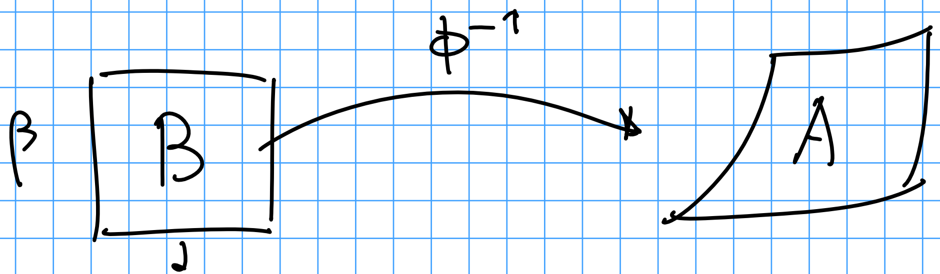
$\frac{4xy}{x^3y^3} - \frac{1}{x^2y^2} = \frac{3}{x^2y^2}$

ALLORA, se applico il teorema direttamente



$\iint_A \det J\phi(x,y)\,dx\,dy = |B| = 1$ NON MI SERVE

Se voglio $|A|$ devo applicare il cambio di variabile contrario



$|A| = \iint_B \det_{\phi^{-1}}(\alpha,\beta)\,d\alpha\,d\beta$

devo invertire $\phi(x,y)$ $\Rightarrow$

$$\begin{cases} y = \alpha x^2 \\ x = \beta y^2 \end{cases}$$

devo ricavare (x,y) in termini di (α,β)

$$\begin{cases} y = \alpha x^2 \\ x = \beta \alpha^2 x^4 \end{cases} \leftarrow \quad x=0 \;/\; 1 = \beta\alpha^2 x^3 \leftrightarrow \boxed{x = \beta^{-1/3}\alpha^{-2/3}}$$

$$y = \alpha\, \beta^{-2/3} \alpha^{-4/3} = \alpha^{-1/3} \beta^{-2/3}$$

$$\Rightarrow \boxed{\phi^{-1}(\alpha,\beta) = \left(\alpha^{-2/3}\beta^{-1/3},\ \alpha^{-1/3}\beta^{-2/3}\right)}$$

$$J\phi^{-1}(\alpha,\beta) = \begin{bmatrix} -\frac{2}{3}\alpha^{-5/3}\beta^{-1/3}, & -\frac{1}{3}\alpha^{-2/3}\beta^{-4/3} \\ -\frac{1}{3}\alpha^{-4/3}\beta^{-2/3}, & -\frac{2}{3}\alpha^{-1/3}\beta^{-5/3} \end{bmatrix}$$

faccio il determinante $\Rightarrow$ $\frac{4}{9}\alpha^{-2}\beta^{-2} - \frac{1}{9}\alpha^{-2}\beta^{-2} = \frac{1}{3}\frac{1}{\alpha^2\beta^2}$

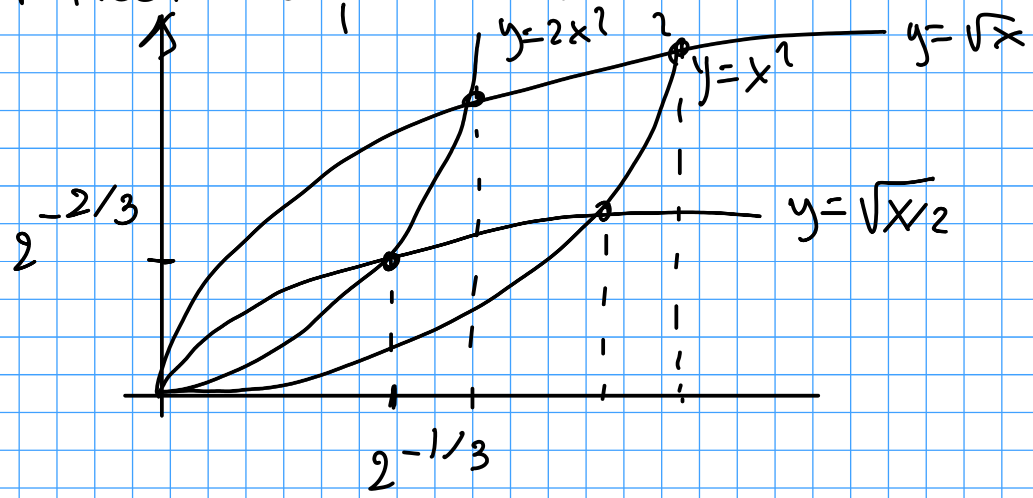
DUNQUE $|A| = \iint_B \frac{d\alpha \, d\beta}{3\alpha^2\beta^2} = \frac{1}{3}\int_1^2 \frac{d\alpha}{\alpha^2}\int_1^2 \frac{d\beta}{\beta^2} =$

$$= \frac{1}{3}\left[-\alpha^{-1}\right]_1^2 \left[-\beta^{-1}\right]_1^2 = \frac{1}{3}\left(1-\frac{1}{2}\right)\left(1-\frac{1}{2}\right) = \boxed{\frac{1}{12}}$$

PER CURIOSITÀ VERIFICHIAMO LA FORMULA (★)

$$\iint_A \frac{3}{x^2y^2}\,dx\,dy = 1 \quad ??$$

POTREI TENTARE DI SPEZZARE A IN "PEZZI NORMALI"



$$\begin{cases} y=\sqrt{x/2} \\ y=x^2 \end{cases} \iff x^4 = \frac{x}{2}$$

$x=0$ NO

$x = 2^{-1/3} = \sqrt[3]{1/2}$

UN PO' LUNGO ...

COORDINATE POLARI (CASO PART. DI CAMBIO DI VAR.)

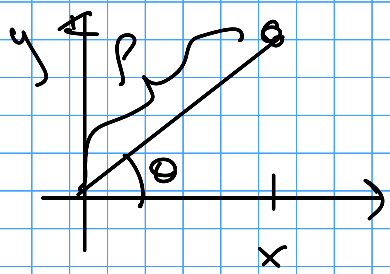
$A \subseteq [0,+\infty[\times [0,2\pi]$ (i punti di A sono coppie (ρ,θ))

Se $(\rho,\theta) \in A$ definisco $\phi(\rho,\theta) = \begin{pmatrix} \rho\cos\theta \\ \rho\sin\theta \end{pmatrix}$

$$J_\phi(\rho,\theta) = \begin{pmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{pmatrix}$$

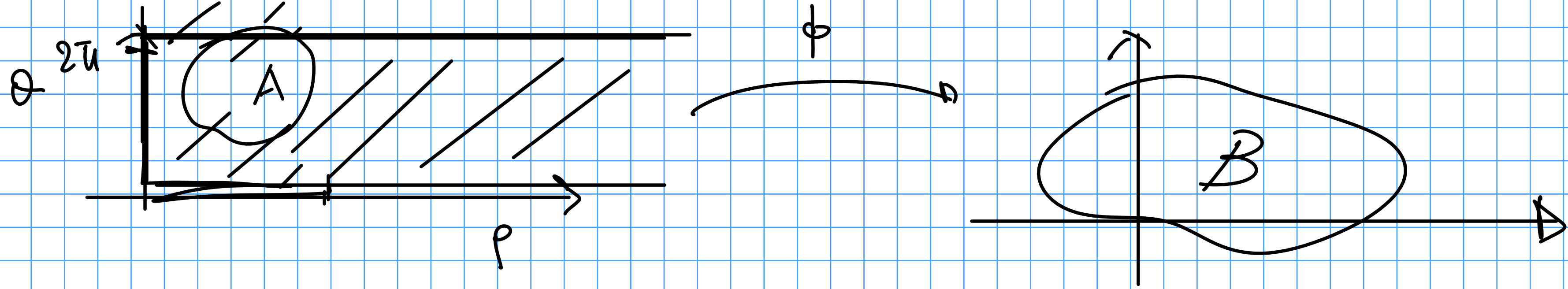
$$\det J_\phi = \rho\cos^2\theta + \rho\sin^2\theta = \rho$$

$$(\rho,\theta) \xrightarrow{\phi}$$



ALLORA se $B = \phi(A)$ e $f : B \to \mathbb{R}$ continua

$$\iint_A f(\rho\cos\theta, \rho\sin\theta)\, \rho \, d\rho\, d\theta = \iint_B f(x,y)\, dx\, dy$$



OSS. IN REALTÀ ϕ NON È BIGETTIVA dato che

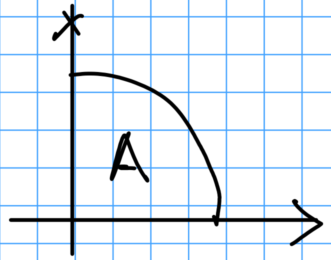
$$\phi(0,\theta) = 0 \quad \forall \theta$$

$$\phi(\rho, 0) = \phi(\rho, 2\pi)$$

PERÒ GLI INSIEMI SU CUI NON È BIGETTIVA SONO TRASCURABILI (segmenti nel piano (ρ, θ)) E LA FORMULA FUNZIONA LO STESSO

ESEMPI DI UTILIZZO DEL CAMBIO DI VAR. IN COORD. POLARI

1) $$\iint_A xy \, dx\,dy \qquad A = \{x^2+y^2 \leq 1,\ x \geq 0,\ y \geq 0\}$$



Se uso le coordinate polari :

$$x = \rho \cos\theta \qquad \text{"} dx\,dy = \rho \, d\rho \, d\theta \text{"}$$
$$y = \rho \sin\theta$$

$$A = \phi(\Omega) \qquad \text{dove } \Omega = \{ 0 \le \rho \le 1 ,\ 0 \le \theta \le \tfrac{\pi}{2} \}$$

$$\Rightarrow \text{INTEGRALE} = \iint_\Omega (\rho\cos\theta)(\rho\sin\theta)\,\rho \, d\rho\, d\theta =$$

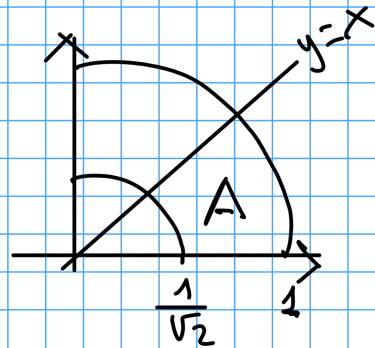
$$\iint_\Omega \sin\theta \cos\theta \ \rho^3 \, d\rho \, d\theta = \left(\int_0^{\pi/2} \sin\theta\cos\theta \, d\theta \right)\left(\int_0^1 \rho^3 d\rho \right) =$$

$$\frac{1}{2}\int_0^{\pi/2} \sin(2\theta)\, d\theta \left[\frac{\rho^4}{4} \right]_0^1 = \frac{1}{2}\left[-\frac{\cos(2\theta)}{2} \right]_0^{\pi/2} \cdot \frac{1}{4} =$$

$$\frac{1}{16}\Big(\underbrace{-\cos(\pi)}_{= -1} + \underbrace{\cos(0)}_{= 1} \Big) = \frac{1}{8}$$

(2) (esercizio di ieri)

$$\iint_A \frac{x}{x^2+y^2}\, dx\, dy$$



$$A = \{ \tfrac{1}{2} \le x^2+y^2 \le 1 ,\ 0 \le y \le x \}$$

Se posso in coord. polari $A = \Phi(\Omega)$ dove

$$\Omega = \left\{ (\rho, \theta) : \frac{\sqrt{2}}{2} \le \rho \le 1 \ , \ 0 \le \theta \le \frac{\pi}{4} \right\}$$

$\Rightarrow$ INTEGRALE $= \displaystyle\iint_{\Omega} \frac{\rho\cos\theta}{\rho^2}\, \rho \, d\rho \, d\theta =$

$$\int_0^{\pi/4} \cos\theta \, d\theta \int_{\frac{\sqrt{2}}{2}}^{1} d\rho = \left(1 - \frac{\sqrt{2}}{2}\right) \Big[\sin\theta \Big]_0^{\pi/4} =$$

$$\frac{2-\sqrt{2}}{2} \ \frac{\sqrt{2}}{2} = \frac{2\sqrt{2}-2}{4} = \frac{\sqrt{2}-1}{2}$$ (TORNA COL RIS. di ieri)

(3) $\displaystyle\iint_{A} \frac{2xy}{(x^2+y^2)(1+x^2+y^2)} \, dx \, dy$ dove

$$A = \{ x \ge 0 , \ y \ge 0 \quad x \le x^2 + y^2 \le 2x \}$$

Se posso in coordinate polari: $x = \rho\cos\theta \quad y = \rho\sin\theta$

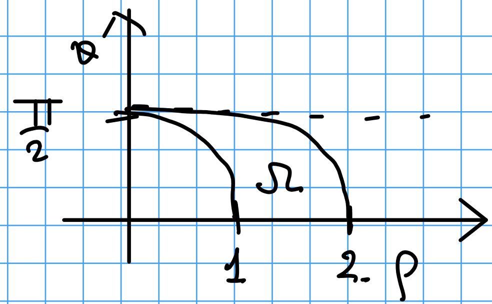
Le condizioni che definiscono A diventano

$$\rho\cos\theta \geq 0 \qquad \rho\sin\theta \geq 0 \qquad (\iff \quad 0 \leq \theta \leq \pi/2)$$

$$\rho\cos\theta \leq \rho^2 \leq 2\rho\cos\theta$$

$$\Updownarrow$$

$$\cos\theta \leq \rho \leq 2\cos\theta$$



$$\Omega = \left\{ 0 \leq \theta \leq \frac{\pi}{2}, \quad \cos\theta \leq \rho \leq 2\cos\theta \right.$$

NORMALE RISPETTO A θ

$$\text{INTEGRALE} = \int_0^{\pi/2} \left(\int_{\cos\theta}^{2\cos\theta} \frac{2\rho^2\cos\theta\sin\theta\ \rho\, d\rho}{\rho^2(1+\rho^2)} \right) d\theta =$$

$$\int_0^{\pi/2} \cos\theta\sin\theta \left(\int_{\cos\theta}^{2\cos\theta} \frac{2\rho}{1+\rho^2}\, d\rho \right) d\theta =$$

$$\int_0^{\pi/2} \cos\theta\sin\theta \left[\ln(1+\rho^2) \right]_{\cos\theta}^{2\cos\theta} d\theta =$$

$$\underbrace{\int_0^{\pi/2} \cos\theta\sin\theta\ln(1+4\cos^2\theta)\,d\theta}_{(I)} - \underbrace{\int_0^{\pi/2} \cos\theta\sin\theta\ln(1+\cos^2\theta)\,d\theta}_{(II)}$$

(I) $\cos^2\theta = t \Rightarrow -2\cos\theta \sin\theta \, d\theta = dt \Leftrightarrow$

$\cos\theta \sin\theta \, d\theta = -\frac{1}{2} dt$

$$\rightarrow -\frac{1}{2}\int_1^0 \ln(1+4t)\,dt$$

$$= \frac{1}{2}\int_0^1 \ln(1+4t)\,dt$$

Stesso discorso per (II). Calcolo

$$I_\alpha = \int_0^1 \ln(1+\alpha t)\,dt = \qquad (\alpha = 4, \ \alpha = 1)$$

$$\left[t\ln(1+\alpha t)\right]_0^1 - \int_0^1 \frac{\alpha t}{1+\alpha t}\,dt =$$

$$\ln(1+\alpha) - \int_0^1 \left(1 - \frac{1}{1+\alpha t}\right) dt = \ln(1+\alpha) - 1 + \int_0^1 \frac{dt}{1+\alpha t} =$$

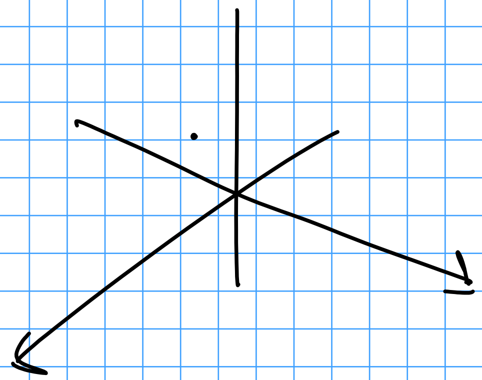
$$\ln(1+\alpha) - 1 + \frac{1}{\alpha}\left[\ln(1+\alpha t)\right]_0^1 = \left(1+\frac{1}{\alpha}\right)\ln(1+\alpha t) - 1$$

$$\Rightarrow \text{INTEGRALE} = \frac{1}{2}(I_4 - I_1) = \frac{1}{2}\left\{\frac{5}{4}\ln(5) \cancel{-1} - 2\ln(2) \cancel{+1}\right\}$$

$$\frac{5}{8}\ln(5) - \ln(2)$$

IN $\mathbb{R}^3$ posso considerare le "coordinate cilindriche"
($\sim$ coordinate polari nel piano x, y)

$$\Phi(\rho, \theta, z) = \begin{pmatrix} \rho\cos\theta \\ \rho\sin\theta \\ z \end{pmatrix}$$



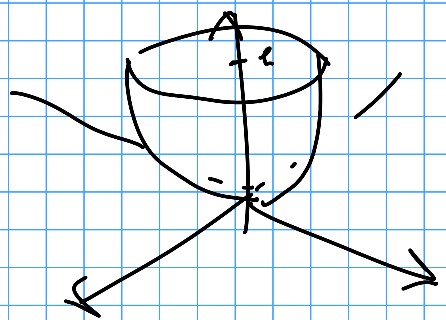
$$J_\Phi = \begin{pmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\det J_\Phi = \rho$ DUNQUE "$dx\,dy\,dz = \rho\,d\rho\,d\theta\,dz$"

ESEMPIO $\displaystyle\iiint_A z\, e^{x^2+y^2}$ $A = \{0 \le z \le 1 \quad x^2+y^2 \le z\}$

$\Updownarrow$

$\rho^2 \le z$



USANDO LE COORD. CILINDRICHE

$$\iiint_A z\, e^{x^2+y^2}\, dx\, dy\, dz = \int_0^1 z\, dz \int_0^{2\pi} d\theta \int_0^{\sqrt{z}} \rho\, e^{\rho^2}\, d\rho =$$

$$2\pi \int_0^1 z\, dz \left[\frac{1}{2} e^{\rho^2} \right]_0^{\sqrt{z}} = 2\pi \int_0^1 \frac{z}{2} \left(e^z - 1\right) dz =$$

$$\pi \int_0^1 z\left(e^z - 1\right) dz = \pi \left[z e^z \right]_0^1 - \pi \int_0^1 e^z\, dz - \pi \int_0^1 z\, dz$$

$$\pi e - \pi(e - 1) - \pi \left[\frac{z^2}{2} \right]_0^1 = \pi - \frac{\pi}{2} = \frac{\pi}{2}$$