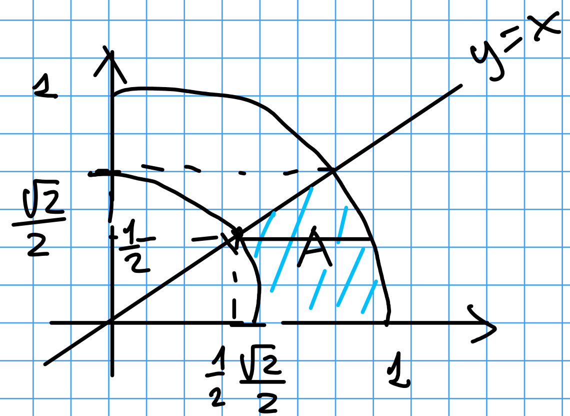
Analisi Matematica II
Lezione 28
1 dicembre 2015

$$\iint_A \frac{x}{x^2+y^2}\,dx\,dy \qquad \text{dove } A=\left\{ \frac{1}{2}\le x^2+y^2\le 1,\ 0\le y\le x \right.$$



A non è normale, né rispetto a x, né rispetto a y

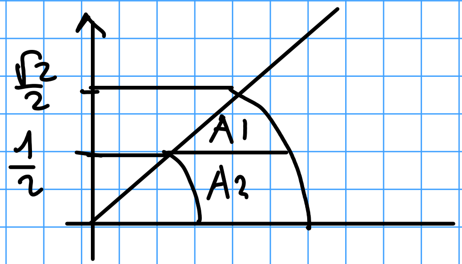
(non posso scrivere $A=\{a\le x\le b,\ \varphi_1(x)\le y\le \varphi_2(x)\}$)

- con φ_1 φ_2 derivabili - lo posso fare se ammetto

che φ_1 φ_2 siano definite a tratti . . .

IL MODO PIÙ SEMPLICE DI DESCRIVERE A

(senza usare le coordinate polari ..) è



$$A_1 = \left\{ \frac{1}{2} \le y \le \frac{\sqrt{2}}{2} , \quad y \le x \le \sqrt{1-y^2} \right\}$$

$$A_2 = \left\{ 0 \le y \le \frac{1}{2} \quad \sqrt{\frac{1}{2}-y^2} \le x \le \sqrt{1-y^2} \right.$$

A_1 e A_2 sono "normali rispetto a y"

$$\iint_{A_1} \frac{x}{x^2+y^2}\,dx\,dy = \int_{1/2}^{\sqrt{2}/2} \left(\int_{y}^{\sqrt{1-y^2}} \frac{x}{x^2+y^2}\,dx \right) dy =$$

$$\int_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}} \left[\frac{1}{2} \ln(x^2+y^2) \right]_{y}^{\sqrt{1-y^2}} dy =$$

$$\int_{1/2}^{\sqrt{2}/2} \left\{ \frac{1}{2} \ln\left(1 \underbrace{-\cancel{y^2}+\cancel{y^2}}_{=0}\right) - \frac{1}{2} \ln\left(y^2+y^2\right) \right\} dy = -\frac{1}{2} \int_{1/2}^{\frac{\sqrt{2}}{2}} \ln(2y^2)\,dy =$$

$$-\frac{1}{2} \int_{1/2}^{\sqrt{2}/2} \ln(2)\,dy - \int_{1/2}^{\sqrt{2}/2} \ln(y)\,dy = -\frac{\ln(2)}{2}\left(\frac{\sqrt{2}}{2}-\frac{1}{2}\right) - \int_{1/2}^{\sqrt{2}/2} \ln(y)\,dy =$$

$$-\frac{\ln(2)}{4}(\sqrt{2}-1)-\Big[y\ln(y)\Big]_{1/2}^{\sqrt{2}/2}+\int_{1/2}^{\sqrt{2}/2} y\,\frac{1}{y}\,dy=-\frac{\ln(2)}{4}(\sqrt{2}-1)+$$

$$-\frac{\sqrt{2}}{2}\ln\frac{\sqrt{2}}{2}+\frac{1}{2}\ln\left(\frac{1}{2}\right)+\frac{\sqrt{2}}{2}-\frac{1}{2}=-\frac{\ln(2)}{4}(\sqrt{2}-1)+\frac{\sqrt{2}}{2}\ln\sqrt{2}-\frac{1}{2}\ln(2)+\frac{\sqrt{2}-1}{2}=$$

$$-\frac{\ln(2)}{4}(\cancel{\sqrt{2}}-1)+\cancel{\frac{\sqrt{2}}{4}\ln(2)}-\frac{1}{2}\ln(2)+\frac{\sqrt{2}-1}{2}=-\frac{\ln(2)}{4}+\frac{\sqrt{2}-1}{2}$$

$$\iint_{A_2}\frac{x}{x^2+y^2}\,dx\,dy=\int_0^{1/2}\left(\int_{\sqrt{\frac{1}{2}-y^2}}^{\sqrt{1-y^2}}\frac{x}{x^2+y^2}\,dx\right)dy=$$

$$\int_0^{1/2}\left[\frac{1}{2}\ln(x^2+y^2)\right]_{\sqrt{1/2-y^2}}^{\sqrt{1-y^2}}dy=\frac{1}{2}\int_0^{1/2}\underbrace{\ln(1-\cancel{y^2}+\cancel{y^2})}_{=0}\,dy$$

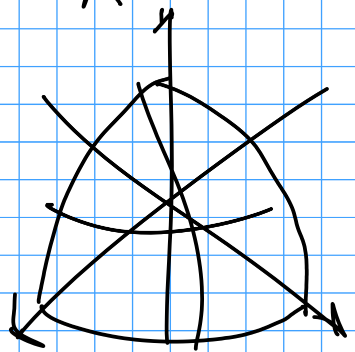
$$-\frac{1}{2}\int_0^{1/2}\ln\left(\frac{1}{2}-\cancel{y^2}+\cancel{y^2}\right)dy=-\frac{1}{2}\int_0^{1/2}\ln\left(\frac{1}{2}\right)dy=\frac{1}{4}\ln(2)$$

$$\Rightarrow\iint_A\frac{x}{x^2+y^2}\,dx\,dy=-\frac{\ln(2)}{4}+\frac{\sqrt{2}-1}{2}+\frac{1}{4}\ln(2)=\frac{\sqrt{2}-1}{2}$$

(LA PROSSIMA VOLTA LO FACCIAMO CON COOR. POLARI)

ALTRO ESERCIZIO

$$\iiint_A xyz \, dx\,dy\,dz \qquad A = \{ x^2+y^2+z^2 \le 1,\ x \ge 0, y \ge 0, z \ge 0 \}$$



Posso vedere A come insieme normale rispetto a z, cioè:

$$A = \{ (x,y) \in A_1,\ \varphi_1(x,y) \le z \le \varphi_2(x,y) \}$$

con $A_1 \subset \mathbb{R}^2$ $\qquad \varphi_1, \varphi_2 : A_1 \to \mathbb{R}$

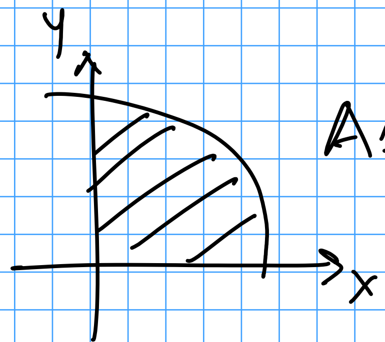
INFATTI VIENE

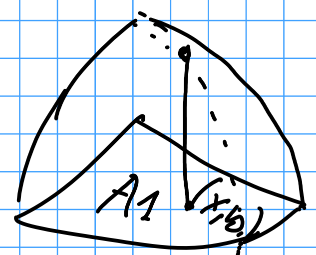
$$A = \{ \underbrace{(x,y) : x^2+y^2 \le 1,\ x \ge 0, y \ge 0}_{(x,y) \in A_1},\ \underset{\varphi_1}{0} \le z \le \underset{\varphi_2}{\sqrt{1-x^2-y^2}} \}$$

USANDO QUESTA RAPPRESENTAZIONE DI A:

$$\iiint_A xyz \, dx\, dy\, dz = \iint_{A_1} \left(\int_0^{\sqrt{1-x^2-y^2}} xyz \, dz \right) dx\, dy =$$

(INTEGRAZIONE "PER FILI")





$$\iint_{A_1} xy \left(\int_0^{\sqrt{1-x^2-y^2}} z \, dz \right) dx\, dy = \iint_{A_1} xy \frac{1}{2} \left[z^2 \right]_0^{\sqrt{1-x^2-y^2}} dx\, dy \rightarrow$$

$$\frac{1}{2} \iint_{A_1} xy(1-x^2-y^2) \, dx\, dy = \frac{1}{2} \iint_{A_1} (xy - x^3y - xy^3) \, dx\, dy$$

A_1 lo descrivo come normale rispetto a x:

$$\frac{1}{2} \int_0^1 \left(\int_0^{\sqrt{1-x^2}} (xy - x^3y - xy^3) \, dy \right) dx \rightarrow$$

$$\frac{1}{2}\int_0^1 \left[x\frac{y^2}{2} - x^3\frac{y^2}{2} - x\frac{y^4}{4}\right]_0^{\sqrt{1-x^2}} dx =$$

$$1-2x^2+x^4$$

$$\frac{1}{8}\int_0^1 \left(2x(1-x^2) - 2x^3(1-x^2) - x(1-x^2)^2\right) dx =$$

$$\frac{1}{8}\int_0^1 \left(2x - 2x^3 - 2x^3 + 2x^5 - x + 2x^3 - x^5\right) dx =$$

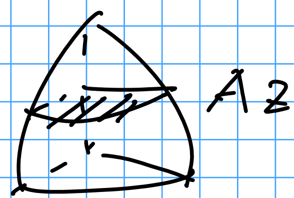
$$\frac{1}{8}\int_0^1 \left(x - 2x^3 + x^5\right) dx = \frac{1}{8}\left[\frac{x^2}{2} - \frac{2x^4}{4} + \frac{x^6}{6}\right]_0^1 =$$

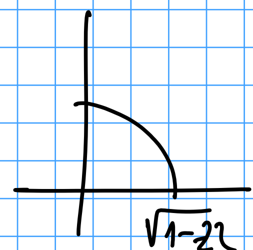
$$\frac{1}{8}\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right) = \frac{1}{48}$$

SI PUÒ PROVARE UN'ALTRA STRADA.

$$A = \{0 \le z \le 1 \,,\ (x,y) \in A_z\}$$

dove $A_z = \{0 \le x^2 + y^2 \le 1 - z^2\}$





(INTEGRO PER SEZIONI)

$$\Rightarrow \iiint_A xyz \, dx\,dy\,dz =$$

$$\int_0^1 z\,dz \iint_{A_z} xy\,dx\,dy = \int_0^1 z\,dz \int_0^{\sqrt{1-z^2}} x\,dx \int_0^{\sqrt{1-x^2-z^2}} y\,dy =$$

$$\int_0^1 z\,dz \int_0^{\sqrt{1-z^2}} x \cdot \frac{1-x^2-z^2}{2}\,dx =$$

$$\frac{1}{2}\int_0^1 z\,dz \int_0^{\sqrt{1-z^2}} (x-x^3-z^2x)\,dx = \frac{1}{2}\int_0^1 z\,dz \left[\frac{x^2}{2}-\frac{x^4}{4}-\frac{z^2x^2}{2}\right]_{x=0}^{x=\sqrt{1-z^2}} =$$

$$\frac{1}{8}\int_0^1 z\left(2(1-z^2)-(1-z^2)^2-2z^2(1-z^2)\right)dz = .$$

- - - Dovrebbe venire lo stesso risultato $\frac{1}{48}$