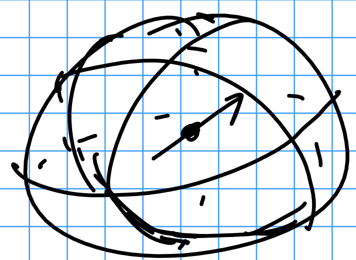
Analisi Matematica II
Lezione 25
24 novembre 2015

PROBLEMA TROVARE MAX / MIN DI

$$f(x,y,z) = z(x^2-y^2) \quad \text{su}$$

$$\Omega := \{ x^2+y^2+z^2 \le 16 \,,\ x+y+z \ge 0 \}$$



- DATO CHE f è continua e Ω è chiuso e limitato

$\exists$ pti di max / min per f su Ω.

LIMITATO : $\Omega \subset B(0,4) = \{ \|(x,y,z)\| \leq 4 \}$

CHIUSO : $\Omega = \underbrace{\{ x^2+y^2+z^2 \leq 16 \}}_{\substack{\Omega_1 \\ \text{chiuso}}} \cap \underbrace{\{ x+y+z \geq 0 \}}_{\substack{\Omega_2 \\ \text{chiuso}}}$

- intersezione di chiusi è chiuso
- In generale un insieme del tipo $\{ \mathbf{x} : G(\mathbf{x}) \leq \text{costante} \}$ con G continua è chiuso

TALI PUNTI DI MAX/MIN $\sqrt{}$ ($\mathbf{x}_0 = (x_0, y_0, z_0)$ di max/min ...) POSSONO :

(a) appartenere a $\mathrm{int}(\Omega) = \{ x^2+y^2+z^2 < 16,\ x+y+z > 0 \}$

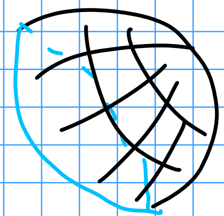
In tal caso sono STAZIONARI : $\nabla f(\mathbf{x}_0) = 0$

(b) appartenere a $\partial\Omega$: ci sono dei "sottocasi"

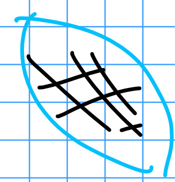
(b1) $\mathbf{x}_0 \in \{ x^2+y^2+z^2 = 16,\ x+y+z > 0 \}$

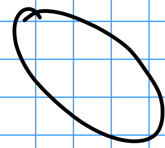
$\mathbf{x}_0$ PTO STAZ. VINCOLATO ; $\exists \lambda \in \mathbb{R}$

$$\nabla f(\mathbf{x}_0) = \lambda_1 \nabla G_1(\mathbf{x}_0)$$



(b2) $\mathbf{x}_0 \in \{ x^2+y^2+z^2 < 16,\ x+y+z = 0 \}$





$$\nabla f(\mathbf{x}_0) = \lambda_2 \nabla G_2(\mathbf{x}_0)$$

(b3) $\mathbf{x}_0 \in \{x^2+y^2+z^2=16,\ x+y+z=0\}$,

$$\Rightarrow \quad \nabla f(\mathbf{x}_0) = \lambda_3 \nabla G_1(\mathbf{x}_0) + \lambda_4 \nabla G_2(\mathbf{x}_0)$$

FORMALMENTE $G_1(x,y,z) = x^2+y^2+z-16$, $G_2(x,y,z) = -x+y+z$

$\Omega = \Omega_1 \cap \Omega_2$ dove $\Omega_1 = \{G_1(x,y,z) \le 0\}$, $\Omega_2 : \{G_2(x,y,z) \le 0\}$

$\Rightarrow \mathrm{int}(\Omega) = \mathrm{int}(\Omega_1) \cap \mathrm{int}(\Omega_2)$ e

$\mathrm{int}(\Omega_1) = \{G_1(x,y,z) < 0\}$, $\mathrm{int}(\Omega_2) = \{G_2(x,y,z) < 0\}$

$$\partial\Omega = \{G_1 = 0\} \cap \{G_2 < 0\} \cup \{G_1 < 0\} \cap \{G_2 = 0\} \cup \{G_1 = 0\} \cap \{G_2 = 0\}$$

$$f(x,y,z) = z(x^2-y^2)$$

(a) Cerco i pti critici "liberi"

$$\frac{\partial f}{\partial x} = 2zx \qquad \frac{\partial f}{\partial y} = -2zy \qquad \frac{\partial f}{\partial z} = x^2-y^2$$

(a) $\begin{cases} 2zx = 0 \\ 2zy = 0 \\ x^2 = y^2 \\ x^2+y^2+z^2 < 16, \; x+y+z > 0 \end{cases} \Rightarrow$ $z = 0$ oppure $x = 0$

$\begin{cases} z = 0 \\ x = \pm y \\ x^2 < 8 \\ x+y > 0 \end{cases} \iff \begin{cases} z = 0 \\ x = y \\ 0 < x < 2\sqrt{2} \end{cases}$ | $\begin{cases} x = 0 \\ y = 0 \\ z^2 < 16 \\ z > 0 \end{cases} \iff \begin{cases} x = 0 \\ y = 0 \\ 0 < z < 4 \end{cases}$

$f(x, x, 0) = 0$ $f(0, 0, z) = 0$

IN TUTTI QUESTI PUNTI f VALE ZERO.

(b1)

$\begin{cases} 2xz = \lambda 2x \\ -2yz = \lambda 2y \\ x^2 - y^2 = \lambda 2z \\ x^2+y^2+z^2 = 16 \\ x+y+z > 0 \end{cases}$

$G_1(x,y,z) = x^2+y^2+z^2-16$

$\nabla G_1 = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\begin{cases} 2x(z-\lambda)=0 \\ 2y(z+\lambda)=0 \\ x^2-y^2=2\lambda z \\ x^2+y^2+z^2=16 \\ x+y+z>0 \end{cases}$$

VARI CASI:

$$(1)\begin{cases} z=\lambda \\ 2y\,2z=0 \\ x^2-y^2=2z^2 \\ x^2+y^2+z^2=16 \\ x+y+z>0 \end{cases} \iff \begin{cases} z=\lambda \\ y=0 \quad // \quad z=0 \\ x^2=y^2+2z^2 \\ x^2+y^2+z^2=16 \\ x+y+z>0 \end{cases}$$

$$(1a)\begin{cases} z=\lambda \\ y=0 \\ x^2=2z^2 \\ 3z^2=16 \\ x+z>0 \end{cases} \iff \begin{cases} z=\lambda=\pm 4/\sqrt{3} \\ x=\pm 4\sqrt{2}/\sqrt{3} \\ x+z>0 \end{cases} \iff \boxed{\begin{matrix} \left(\frac{4\sqrt{6}}{3}, 0, \frac{4}{3}\sqrt{3}\right) \\ \left(\frac{4\sqrt{6}}{3}, 0, -\frac{4}{3}\sqrt{3}\right) \end{matrix}}$$

$\left(\lambda=\frac{4}{3}\sqrt{3}\right)$

$\left(\lambda=-\frac{4}{3}\sqrt{3}\right)$

$$(1b)\begin{cases} \lambda=z=0 \\ x^2=y^2=8 \\ x+y>0 \end{cases}$$

$\dashrightarrow$ PTI. CRITICI LIBERI, f VALE ZERO

$$(2)\begin{cases} z=-\lambda \\ 2x(2z)=0 \\ y^2=x^2+2z^2 \\ x^2+y^2+z^2=16 \\ x+y+z>0 \end{cases} \qquad (2a)\begin{cases} z=-\lambda \\ x=0 \\ y^2=2z^2 \\ x^2+y^2+z^2=16 \\ x+y+z>0 \end{cases} \iff \begin{cases} \lambda=-z \\ x=0 \\ y=\pm\frac{4}{3}\sqrt{6} \\ z=\pm\frac{4}{3}\sqrt{3} \\ z+y>0 \end{cases} \iff \begin{matrix} \left(0, \frac{4\sqrt{6}}{3}, \frac{4\sqrt{3}}{3}\right) \\ \left(0, \frac{4\sqrt{6}}{3}, -\frac{4\sqrt{3}}{3}\right) \end{matrix}$$

(2b) $z=0=\lambda$ - come sopra f vale zero

$$f\left(\frac{4\sqrt6}{3},0,\frac{4\sqrt3}{3}\right)=\frac{4\sqrt3}{3}\,\frac{16\cdot6}{3}=\frac{128}{3}\sqrt3\;;\; f\left(\frac{4\sqrt6}{3},0,\frac{-4\sqrt3}{3}\right)=-\frac{128\sqrt3}{3}$$

$$f\left(0,\frac{4\sqrt6}{3},\frac{4\sqrt3}{3}\right)=-\frac{4\sqrt3}{3}\,\frac{16\cdot6}{3}=-\frac{128}{3}\sqrt3\;;\; f\left(0,\frac{4\sqrt6}{3},\frac{-4\sqrt3}{3}\right)=\frac{128\sqrt3}{3}$$

(b2) $\begin{cases} 2xz=\lambda \\ -2yz=\lambda \\ x^2-y^2=\lambda \\ x+y+z=0 \\ x^2+y^2+z^2<16 \end{cases} \iff \begin{cases} \lambda=2xz \\ 2xz+2yz=0 \\ x^2-y^2=2xz \\ x+y+z=0 \\ x^2+y^2+z^2<16 \end{cases} \iff \begin{cases} \lambda=2xz \\ z=0 \;//\; x+y=0 \\ x^2-y^2=2xz \\ x+y+z=0 \\ x^2+y^2+z \end{cases}$

IN OGNI CASO TROVO $z=0$: se infatti $x+y=0$ dalla IV segue $z=0$

MA $z=0$ CORRISPONDE A CASI GIÀ CONSIDERATI ($\lambda=0$) per cui

$f=0$

(b3) $\begin{cases} 2xz=2\lambda x+\mu \\ -2yz=2\lambda y+\mu \\ x^2-y^2=2\lambda z+\mu \\ x^2+y^2+z^2=16 \\ x+y+z=0 \end{cases} \Rightarrow \begin{cases} 2x^2z=2\lambda x^2+\mu x \\ -2y^2z=2\lambda y^2+\mu y \\ z(x^2-y^2)=2\lambda y^2+\mu z \\ x^2+y^2+z^2=16 \\ x+y+z=0 \end{cases}$ SOMMO

$$\Rightarrow \quad 2x^2 z - 2y^2 z + z(x^2-y^2) = 2\lambda \underbrace{(x^2+y^2+z^2}_{=16} + \mu \underbrace{(x+y+z}_{=0})$$

$$\Rightarrow \quad \lambda = \frac{1}{32}\left(2x^2 z - 2y^2 z + z x^2 - z y^2\right) = \frac{3z}{32}\left(x^2-y^2\right) =$$

$$\frac{3z}{32}(x-y)(x+y) = -\frac{3}{32} z^2 (x-y) \qquad \left[z = -(x+y)\right]$$

SE INVECE SOMMO LE PRIME TRE RIGHE IN ⍟

$$3\mu + 2\lambda \underbrace{(x+y+z)}_{=0} = 2xz - 2yz + x^2 - y^2 \Leftrightarrow$$

$$3\mu = 2z(x-y) + (x-y)\underbrace{(x+y)}_{=-z} = (x-y)(2z-z) \Rightarrow$$

$$\mu = \frac{(x-y)z}{3} \qquad \lambda = -\frac{3}{32} z^2 (x-y)$$

METTIAMO QUESTI VALORI IN ⍟

$$\begin{cases} 2xz = -\frac{3}{16} z^2 x (x-y) + \frac{z}{3}(x-y) \\ -2yz = -\frac{3}{16} z^2 y (x-y) + \frac{z}{3}(x-y) \\ -z(x-y) = -\frac{3}{16} z^3 (x-y) + \frac{z}{3}(x-y) \\ x^2+y^2+z^2 = 16 \\ x+y+z = 0 \end{cases}$$

NELLA III^a RIGA ABBIAMO $x = y$ oppure $-z = -\frac{3}{16} z^3 + \frac{z}{3}$

Nel primo caso, si ottiene $z = -2x$ (dall'ultima) e dalla I^o

$-4x^2 = 0 \implies x=0, y=0, z=0$ IMPOSSIBILE

Nel secondo caso abbiamo $\frac{9z^3}{16} = 4z \iff z=0 \,//\, z^2 = \frac{16 \cdot 4}{9}$

Escludiamo $z=0$ (in questo caso f fa zero) $\implies z = \pm \frac{8}{3}$

Mettendo in (A) questo valore:

$$\begin{cases} 2x = \left(\mp \frac{1}{2} x + \frac{1}{3}\right)(x-y) \\ -2y = \left(\mp \frac{1}{2} y + \frac{1}{3}\right)(x-y) \\ z = \pm 8/3 \\ x+y = \mp \frac{8}{3},\ x^2+y^2 = \frac{144}{9} \end{cases}$$

(SOMMO I e II) $\implies$

$z = 8/3 \quad x+y = -\frac{8}{3}$

$x = \frac{4}{3} = y$

$y = x \mp \frac{8}{3}$

$y^2 = x^2 + \frac{64}{9} \mp \frac{16}{3} x$

$2x^2 \mp \frac{16}{3} x + \frac{64}{9} - \frac{144}{9}$

$$\Rightarrow \quad 2x^2z - 2y^2z + z(x^2-y^2) = 2\lambda\underbrace{(x^2+y^2+z^2)}_{=16} + \mu\underbrace{(x+y+z)}_{=0}$$

$$\Rightarrow \quad \lambda = \frac{1}{32}\left(2x^2z - 2y^2z + zx^2 - zy^2\right) = \frac{3z}{32}\left(x^2-y^2\right).$$

METTENDO NEL SISTEMA L'ESPRESSIONE DI λ

$$\begin{cases} 2xz = \dfrac{3xz}{16}(x^2-y^2) + \mu \\ -2yz = \dfrac{3yz}{16}(x^2-y^2) + \mu \\ x^2-y^2 = \dfrac{3z^2}{16}(x^2-y^2) + \mu \\ x^2+y^2+z^2 = 16 \\ x+y+z = 0 \end{cases} \iff \begin{cases} \mu = xz\left(2 - \dfrac{3}{16}(x^2-y^2)\right) \\ 2z(x+y) = \dfrac{3z}{16}(x-y)(x^2-y^2) \\ x^2-y^2+2yz = \dfrac{3z}{16}(z-y)(x^2-y^2) \\ x^2+y^2+z^2 = 16 \\ x+y+z = 0 \end{cases}$$

$(x+y)$

$(x-y)$

$$(1)\ \begin{cases} \mu = \dots \\ x+y = 0 \\ \vdots \\ z = 0 \end{cases} \iff \begin{cases} \mu = \dots \\ z = 0 \\ x = -y \\ 2x^2 = 16 \end{cases} \iff \pm(2\sqrt{2}, -2\sqrt{2}, 0)$$

IN OGNI CASO f fa ZERO

$$(2)\begin{cases} u = \dots \\ z = 0 \quad \text{— — —} \\ \vdots \end{cases}$$

f lo ze

$$(3)\begin{cases} u = \dots \\ 3z = (x-y)^2 \\ 32yz = [3z(z-y) - 16](x^2-y^2) \\ x^2+y^2+z^2 = 16 \\ x+y+z=0 \end{cases}$$

$$\begin{cases} u = \dots \\ x - y = \pm 4\sqrt{2} \\ 32xy = 4\sqrt{2}(x+y)(3z(z-y)-16) \\ x^2+y^2+z^2 = 16 \\ x+y+z=0 \end{cases}$$

$$\Leftrightarrow \begin{cases} u = \dots \\ x = \pm 4\sqrt{2} + y \\ z = -2y \pm 4\sqrt{2} \\ x^2+y^2+z^2 = 16 \\ 3y(y \pm 4\sqrt{2}) = 3\dots \end{cases}$$