Analisi Matematica II
Lezione 24
23 novembre 2015

ESERCIZI

Consideriamo

$$f(x,y) := 8xy + x^4 + 16y^4$$

$$M := \{(x,y) : f(x,y) = 52\}$$

(a) Studiare i pti staz. di f.

$$\frac{\partial}{\partial x} f = 8y + 4x^3 \qquad \frac{\partial f}{\partial y} = 8x + 64y^3$$

$$8^3 = \left(2^3\right)^3 = 2^9$$

$$\begin{cases} 2y + x^3 = 0 \\ x + 8y^3 = 0 \end{cases} \quad \begin{cases} 2y - 8^3 y^9 = 0 \\ x = -8y^3 \end{cases} \quad \begin{cases} y = 0 \\ x = 0 \end{cases} \text{ oppure } \begin{cases} 2^8 y^8 = 1 \\ x = -8y^3 \end{cases}$$

$$\begin{cases} y = \pm 1/2 \\ x = \mp 1 \end{cases}$$

$\Rightarrow$ TRE PTI STAZ.: $(0,0)$ $(1,-1/2)$ $(-1,1/2)$

Classifico i pti staz. usando l'Hessiano:

$$\frac{\partial^2 f}{\partial x^2} = 12x^2 \qquad \frac{\partial^2 f}{\partial y^2} = 64 \cdot 3y^2 \qquad \frac{\partial^2 f}{\partial x \partial y} = 8$$

- $Hf(0,0) = \begin{pmatrix} 0 & 8 \\ 8 & 0 \end{pmatrix}$ $\quad \det = -8^2 < 0 \quad$ SELLA

- $Hf(1,-1/2) =$
  $Hf(-1,1/2) = \begin{pmatrix} 12 & 8 \\ 8 & 48 \end{pmatrix}$ $\quad \det > 0, \quad 12 > 0$

  MINIMI

- NOTA $\boxed{\lim_{\|x,y\| \to \infty} f(x,y) = +\infty}$ $\qquad |xy| \leq \frac{x^2+y^2}{2}$

$$\Downarrow$$

$$f(x,y) \geq \underbrace{-\frac{x^2}{2} + x^4}_{(1)} \underbrace{-\frac{y^2}{2} + 16y^4}_{(2)} \qquad xy \geq -\frac{x^2+y^2}{2}$$

Se $\|(x,y)\| \to \infty$ almeno uno tra $|x|$ e $|y|$ diverge

$\Rightarrow$ uno almeno tra (1) e (2) tende a $+\infty$ (e entrambi sono limitati inferiormente) -

- DIRE SE M è "un dominio regolare" (DOPO ...)

- DIRE SE M è localmente grafico di una funzione $y = g(x)$ VICINO AL PUNTO $P = (\sqrt{2}, -\sqrt{2})$

NOTA $P \in M$ dato che $f(\sqrt{2}, -\sqrt{2}) = 8(-2) + 4 + 16 \cdot 4$
$= -16 + 4 + 64 = 52$ (VERIFICA la condizione di appartenenza ad M).

- Per vedere se esiste una tale $g(x)$

calcolo $\frac{\partial f}{\partial y} = 8x + 64 y^3 \Big|_{\substack{x=\sqrt{2} \\ y=-\sqrt{2}}} = 8\sqrt{2} - 128\sqrt{2} = -120\sqrt{2} < 0$

APPLICO DINI $\Rightarrow$ M è grafico di una $g(x)$ per $x \simeq \sqrt{2}$
(con $g(\sqrt{2}) = -\sqrt{2}$)

DOMANDA Calcolare $g'(\sqrt{2})$ e $g''(\sqrt{2})$

Dall'enunciato del Dini:

$$g'(\sqrt{2}) = \frac{\frac{\partial}{\partial x} f(\sqrt{2}, -\sqrt{2})}{-\frac{\partial f}{\partial y}(\sqrt{2}, \sqrt{2})} = \frac{8(-\sqrt{2}) + 4 \, 2\sqrt{2}}{-(8\sqrt{2} - 64 \cdot 2\sqrt{2})}$$

$$= \frac{-8\sqrt{2} + 8\sqrt{2}}{120\sqrt{2}} = 0$$

Per calcolare g'' posso usare la formula

$$g'(x) = -\frac{\frac{\partial}{\partial x} f(x, g(x))}{\frac{\partial}{\partial y} f(x, g(x))} \Rightarrow \text{(DERIVO)}$$

$$-\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y}$$

$$- \frac{\left[\frac{\partial^2}{\partial x^2} f(x, g(x)) \cdot 1 + \frac{\partial^2}{\partial x \partial y} f(x, g(x))\, g'(x)\right] \frac{\partial}{\partial y} f(x, g(x))}{\left(\frac{\partial}{\partial y} f(x, g(x))\right)^2} +$$

$$+ \frac{\frac{\partial}{\partial x} f(x, g(x)) \left[\frac{\partial^2}{\partial x \partial y} f(x, g(x)) \cdot 1 + \frac{\partial^2}{\partial y^2} f(x, g(x))\, g'(x)\right]}{\left(\frac{\partial}{\partial y} f(x, g(x))\right)^2} =$$

$$\frac{-\frac{\partial^2}{\partial x^2} f \cdot \frac{\partial f}{\partial y} + 2 \frac{\partial^2 f}{\partial x \partial y} \frac{\partial f}{\partial x} - \frac{\partial^2}{\partial y^2} f \left(\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y}\right) \frac{\partial f}{\partial x}}{\left(\frac{\partial}{\partial y} f(x, g(x))\right)^2} =$$

$$\frac{-\frac{\partial^2}{\partial x^2} f \left(\frac{\partial f}{\partial y}\right)^2 + 2\frac{\partial^2 f}{\partial x \partial y}\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial^2}{\partial y^2}\left(\frac{\partial f}{\partial x}\right)^2}{\left(\frac{\partial}{\partial y} f\right)^3} \qquad \left(= g''(x)\right)$$

(FORMULA GENERICA)

Nel nostro caso

$\left(\frac{\partial f}{\partial x}(\sqrt{2}, -\sqrt{2}) = 0\right)$

$$g''(\sqrt{2}) = \frac{-(120\sqrt{2})^2 \cdot 12 \cdot 2 + 0 + 0}{(-120 \cdot \sqrt{2})^3}$$

$$= \frac{24}{120\sqrt{2}} = \frac{1}{5\sqrt{2}} = \frac{\sqrt{2}}{10}$$

(NOTA $g'(\sqrt{2}) = 0$

$g''(\sqrt{2}) > 0$

$\Rightarrow \sqrt{2}$ è pto di min locale per g.)

OSS L'espressione di $g''(x)$ si può anche ottenere da

$f(x, g(x)) = 52 \quad \forall x \simeq \sqrt{2}$: SE DERIVO:

$$\frac{\partial f}{\partial x}(x, g(x)) + \frac{\partial f}{\partial y}(x, g(x))\, g'(x) = 0 \qquad \text{RI-DERIVO}$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x \partial y} g'(x) + \frac{\partial^2 f}{\partial x \partial y} g'(x) + \frac{\partial^2 f}{\partial^2 y^2} g'(x)^2 + \frac{\partial f}{\partial y} g''(x) =$$

$$\frac{\partial^2 f}{\partial x^2} + 2\frac{\partial^2 f}{\partial x \partial y} g' + \frac{\partial^2 f}{\partial y^2} (g')^2 + \frac{\partial f}{\partial y} g'' \Longrightarrow$$

$$g''(x) = -\frac{1}{\frac{\partial f}{\partial y}(x, g(x))} \left(\frac{\partial^2 f}{\partial x^2}(x, g(x)) + 2\frac{\partial^2 f(x, g(x))}{\partial x \partial y} g'(x) + \frac{\partial^2 f(x, g(x))}{\partial y^2} g'(x)^2 \right)$$

$$= -\frac{1}{\frac{\partial f}{\partial y}(x, g(x))} \left(\frac{\partial^2 f}{\partial x^2}(x, g(x)) - 2\frac{\partial^2 f}{\partial x \partial y}(x, g(x)) \frac{\frac{\partial f}{\partial x}(x, g(x))}{\frac{\partial f}{\partial y}(x, g(x))} + \frac{\partial^2 f}{\partial y^2}(x, g(x)) \frac{\frac{\partial f}{\partial x}(x, g(x))^2}{\frac{\partial f}{\partial y}(x, g(x))^2} \right)$$

$$= \frac{-1}{\left(\frac{\partial f}{\partial y}(x, g(x))\right)^3} \left(\frac{\partial^2 f}{\partial x^2}(x, g(x)) \frac{\partial f}{\partial y}(x, g(x))^2 - 2\frac{\partial^2 f}{\partial x \partial y}(x, g(x)) \frac{\partial f}{\partial x}(x, g(x)) \frac{\partial f}{\partial y}(x, g(x)) + \frac{\partial^2 f}{\partial y^2}(x, g(x)) \frac{\partial f}{\partial x}(x, g(x))^2 \right)$$

ESERCIZIO VINCOLI DATI DA PIÙ EQUAZIONI

$$M = \{ (x,y,z,w) : \underbrace{x^2 + y^2 + z^2 + w^4 = 1}_{G_1}, \underbrace{x + 2y + 3z + 4w = 2}_{G_2} \}$$

VOGLIO VEDERE "COSA SUCCEDE" VICINO AL PUNTO $(0,1,0,0)$

che si trova in M — VORREI RICAVARE DUE VARIABILI IN FUNZIONE DELLE ALTRE DUE.

APPLICO DINI (GENERALE): calcolo la Jacobiana di

$$G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} \qquad (\text{da } \mathbb{R}^4 \to \mathbb{R}^2)$$

$$J_G(x,y,z,w) = \begin{pmatrix} \frac{\partial G_1}{\partial x} & \frac{\partial G_1}{\partial y} & \frac{\partial G_1}{\partial z} & \frac{\partial G_1}{\partial w} \\ \frac{\partial G_2}{\partial x} & \frac{\partial G_2}{\partial y} & \frac{\partial G_2}{\partial z} & \frac{\partial G_2}{\partial w} \end{pmatrix} =$$

$$\left.\begin{pmatrix} 2x, & 2y, & 2z, & 4w^3 \\ 1, & 2, & 3, & 4 \end{pmatrix}\right|_{(0,1,0,0)} =$$

$$\begin{pmatrix} 0, & \boxed{\begin{matrix} 2, & 0 \\ 2, & 3 \end{matrix}} & , 0 \\ 1, & & , 4 \end{pmatrix}$$

OK $(\det = 6 \neq 0)$

NON VA BENE

NON TROVO $z = \varphi(x,y)$

$w = \psi(x,y)$

$\Rightarrow$ Esistono due funzioni φ_1 e φ_2 tali che — su M –

$y = \varphi_1(x, w)$ $\quad z = \varphi_2(x, w)$ $\quad$ [$M = \{(x, \varphi_1(x,w), \varphi_2(x,w), w)\}$ vicino a $(0,1,0,0)$.]

- NATURALMENTE $\varphi_1(0,0) = 1$ $\quad \varphi_2(0,0) = 0$

(perché $(0,1,0,0) \in M$)

- VOGLIO CALCOLARE $\frac{\partial}{\partial x}\varphi_1(0,0)$ $\quad \frac{\partial}{\partial w}\varphi(0,0)$

$\frac{\partial}{\partial x}\varphi_2(0,0)$ $\quad \frac{\partial}{\partial w}\varphi_2(0,0)$

DINI AFFERMA

$$\left(\Phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \right)$$

$$J_{\Phi}(0,0) = -\left(\frac{\partial G}{\partial(y,z)}\right)^{-1} \frac{\partial G}{\partial(x\,w)} \qquad \begin{pmatrix} x=0 & y=1 \\ z=0 & w=0 \end{pmatrix}$$

$$- \begin{bmatrix} 2 & 0 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 2 & 3 \end{bmatrix}^{-1} = \frac{1}{\underset{\substack{\uparrow \\ \det}}{6}} \begin{bmatrix} 3 & -2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/3 \\ 0 & 1/3 \end{bmatrix}$$

$$\Rightarrow J\Phi(0,0) = \begin{bmatrix} -1/2 & 1/3 \\ 0 & -1/3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1/3 & 4/3 \\ -1/3 & -4/3 \end{bmatrix}$$

CIOÈ

$$\frac{\partial \varphi_1}{\partial x} = 1/3 \qquad \frac{\partial \varphi_1}{\partial w} = \frac{4}{3}$$

$$\frac{\partial \varphi_2}{\partial x} = -\frac{1}{3} \qquad \frac{\partial \varphi_2}{\partial w} = -\frac{4}{3}$$

!!

(esercizio 11 pag 206 ADAMS) La versione del libro chiede di esplicitare y e w in termini di x e z

CIÒ È POSSIBILE DATO CHE

$$\begin{bmatrix} 0 & 2 & 0 & 6 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 2 & 4 \end{bmatrix}$$ ha det $8 \neq 0$

... (RIFARE TUTTO)

ALTRO ESEMPIO

$$M = \{ \underbrace{xyzw = 1}_{G_1} ; \underbrace{x+y+z+w = 0}_{G_2} \} \qquad G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$$

$$(1,-1,1,-1) \in M$$

$$J_G = \begin{pmatrix} yzw, & xzw, & xyw, & xyz \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$J_G(1,-1,1,-1) = \begin{array}{c} \begin{pmatrix} 1 & \boxed{\begin{matrix} -1 & 1 \\ 1 & 1 \end{matrix}} & -1 \\ 1 & & 1 \end{pmatrix} \\ \begin{matrix} x \quad & y \quad & z \quad & w \end{matrix} \end{array}$$

(esercizio o pag 206 dell'ADAMS) DOMANDA posso descrivere M mediante $y = \varphi_1(x, w)$ $z = \varphi_2(x, w)$

PER FARLO MI SERVE CHE

$$\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$ INVERTIBILE $\boxed{\text{SÌ}}$ $\det = -2$

NOTA CHE L'INVERSA È $\dfrac{1}{-2}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \leftarrow$

INOLTRE $\left(\text{se } \Phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \right)$ $J_\Phi = -\left[\dfrac{\partial G}{\partial(y,z)}\right]^{-1} \left[\dfrac{\partial G}{\partial(x,w)}\right]$

$$= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

CIOÈ

$$\frac{\partial \varphi_1}{\partial x} = 0 \qquad \frac{\partial \varphi_1}{\partial w} = -1$$

$$\frac{\partial \varphi_2}{\partial x} = -1 \qquad \frac{\partial \varphi_2}{\partial w} = 0$$

PROBLEMA

TROVARE MAX / MIN DI

$$f(x,y,z) = z(x^2-y^2) \quad \text{su}$$

$$\Omega := \{ x^2+y^2+z^2 \le 16 ,\ x+y+z \ge 0 \}$$