Analisi Matematica II

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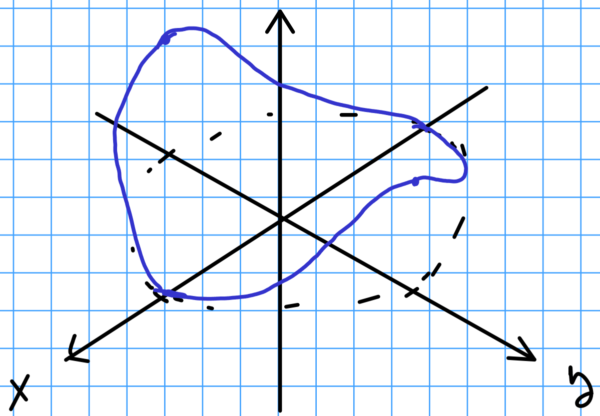
ricevimento: il venerdì alle 11.00 - via Buonarroti 1/c,
oppure su appuntamento (da concordare via email)

ESERCIZIO. Calcolare $\int_\gamma \vec{F} \cdot d\vec{s}$ dove $\vec{F} = \begin{pmatrix} y \\ -x \\ z^2 \end{pmatrix}$ e

γ descrive $\{ z = y^2,\ x^2+y^2 = 4 \}$ con verso antiorario visto da sopra



MODO DIRETTO :

$\gamma(t) = (2\cos(t),\ 2\sin(t),\ 4\sin^2(t))$ $\quad 0 \leq t \leq 2\pi$

$\gamma'(t) = (-2\sin(t),\ 2\cos(t),\ 8\sin(t)\cos(t))$

$$\int_\gamma \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \begin{pmatrix} 2\sin(t) \\ -2\cos(t) \\ 16\sin^4(t) \end{pmatrix} \begin{pmatrix} -2\sin(t) \\ 2\cos(t) \\ 8\sin(t)\cos(t) \end{pmatrix} dt = \int_0^{2\pi} -4 + 128\sin^5(t)\cos(t) =$$

$$-8\pi + 128 \left[\frac{\sin^6(t)}{6} \right]_0^{2\pi} = -8\pi$$

USANDO STOKES : Introduco $S = T(B)$ dove $B = \{ u^2+v^2 \leq 4 \}$ e $T(u,v) = \begin{pmatrix} u \\ v \\ v^2 \end{pmatrix}$

da cui γ descrive il bordo di S e $T_u \otimes T_v = \begin{pmatrix} 0 \\ -2v \\ 1 \end{pmatrix}$. Inoltre

$$\vec{f} := \mathrm{rot}\begin{pmatrix} y \\ -x \\ z^2 \end{pmatrix} = \left(\frac{\partial}{\partial y} z^2 - \frac{\partial}{\partial z}(-x)\right) i + \left(\frac{\partial}{\partial z} y - \frac{\partial}{\partial x} z^2\right) j + \left(\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y} y\right) = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} \Rightarrow$$

$$\int_\gamma \vec{F} \cdot d\vec{s} = \iint_S \vec{f} \cdot \vec{\nu}\, d\sigma = \iint_B \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} \begin{pmatrix} 0 \\ -2v \\ 1 \end{pmatrix} du\, dv = -2 \iint_B du\, dv = -2\pi\, 2^2 = -8\pi$$

• **ESERCIZIO** Calcolare $\int_\gamma \vec{F} \cdot d\vec{s}$ dove $\vec{F} = \begin{pmatrix} y e^x \\ x^2 + e^x \\ z^2 e^z \end{pmatrix}$ e

$$\gamma(t) = (1+\cos(t),\ 1+\sin(t),\ 1-\sin(t)-\cos(t)) \qquad 0 \le t \le 2\pi$$

DIRETTAMENTE $\gamma' = (-\sin(t),\ \cos(t),\ \cos(t)-\sin(t)) \Rightarrow$

$$\int_\gamma \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \begin{pmatrix} (1+\sin(t))\, e^{1+\cos(t)} \\ (1+\cos(t))^2 + e^{1+\cos(t)} \\ (1-\sin(t)-\cos(t))^2\, e^{1-\cos(t)-\sin(t)} \end{pmatrix} \begin{pmatrix} -\sin(t) \\ \cos(t) \\ \cos(t)-\sin(t) \end{pmatrix} = ??$$

USANDO STOKES Noto che γ è il bordo di S dove

$$S = \{ (x-1)^2 + (y-1)^2 \le 1,\ x+y+z = 3 \}$$ che si può parametrizzare

con $\Gamma(u,v) = \begin{pmatrix} u \\ v \\ 3-u-v \end{pmatrix}$ per $(u,v) \in D := \{ (u-1)^2 + (v-1)^2 \le 1 \} \Rightarrow$

$$\Gamma_u \otimes \Gamma_v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

. Inoltre se $\vec{f} := \mathrm{rot}(\vec{F})$ si ha:

$$\vec{g} = \begin{pmatrix} 0 \\ 0 \\ 2x + e^{x} - e^{x} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2x \end{pmatrix} \Longrightarrow$$

$$\int_{\gamma} \vec{F} \cdot d\vec{s} = \iint_{S} \vec{g} \cdot \vec{\nu} \, d\sigma = \iint_{D} \begin{pmatrix} 0 \\ 0 \\ 2u \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} du \, dv = \iint_{D} 2u \, du \, dv =$$

$$\left(u = (1 + \rho \cos\theta) \, , \quad v = (1 + \rho \sin\theta) \qquad du \, dv = \rho \, d\rho \, d\theta \right) =$$

$$\int_0^{2\pi} \left(\int_0^1 2(1 + \rho \cos\theta) \rho \, d\rho \right) d\theta = \int_0^{2\pi} \left(\int_0^1 2\rho \, d\rho \right) d\theta + \int_0^{2\pi} \left(\int_0^1 2\rho^2 \, d\rho \right) \cos\theta \, d\theta =$$

$$2\pi \left[\rho^2 \right]_0^1 + \underbrace{\int_0^{2\pi} \cos\theta \, d\theta}_{=0} \left[\frac{2}{3} \rho^3 \right]_0^1 = 2\pi$$