Analisi Matematica II

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oppure su appuntamento (da concordare via email)

RIPRENDO L'ESERCIZIO:

$$\iint_S \vec{g}\cdot\vec{\nu}\, d\sigma \quad \text{dove} \quad \vec{g}(x,y,z) = \begin{pmatrix} -2y+2x \\ -2x \\ -2z-3 \end{pmatrix}$$

S (= semisfera) $= \{x^2+y^2+z^2=1,\ z\geq 0\}$

VEDIAMO SE SI PUÒ USARE STOKES - se $\exists\, \vec{F}$ tale che

$\mathrm{rot}(\vec{F}) = \vec{g}$. Questo è vero dato $\mathrm{div}(\vec{g})=0$

$$\left(\mathrm{div}\,\vec{g} = \frac{\partial}{\partial x}(-2y+2x) + \frac{\partial}{\partial y}(-2x) + \frac{\partial}{\partial z}(-2z-3) = 2-2=0 \right)$$

COME TROVO $\vec{F}$ (so che esiste)

OSS. Se $\mathrm{rot}\vec{F} = \vec{g}$ allora anche $\vec{F_1} = \vec{F} + \nabla\phi$ (con ϕ scalare) verifica $\mathrm{rot}\,\vec{F_1} = \vec{g}$, infatti

$$\mathrm{rot}(\vec{F}+\nabla\phi) = \underbrace{\mathrm{rot}(\vec{F})}_{=\vec{g}} + \underbrace{\mathrm{rot}\,\nabla\phi}_{=0} = \vec{g}$$

- IN EFFETTI, almeno su Ω semplicemente connesso, vale $\Longleftrightarrow$

se $\mathrm{rot}\,\vec{F} = \mathrm{rot}\,\vec{F_1} \Longleftrightarrow \mathrm{rot}(\vec{F}-\vec{F_1})=0 \Longleftrightarrow \vec{F}-\vec{F_1} = \nabla\phi$

per un opportuno ϕ

DUNQUE le sol. di $rot\,\vec{F} = \vec{g}$ sono tutte e sole del tipo $\vec{F}_0 + \nabla \phi$ dove ϕ è scalare e $\vec{F}_0$ è una sol. particolare (se siamo su Ω semplicemente connesso)

• QUESTO ULTIMO DISCORSO MI DICE CHE POSSO TROVARE $\vec{F}$ tale che $rot\,\vec{F} = \vec{g}$ CON LA PROPRIETÀ $\vec{F} = \begin{pmatrix} F_1 \\ F_2 \\ 0 \end{pmatrix}$

INFATTI SE $\vec{F} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$ (e $rot\,\vec{F} = \vec{g}$) POSSO TROVARE ϕ tale che $\frac{\partial}{\partial z}\phi = F_3$: basta prendere

$$\phi(x, y, z) = \int_0^z F_3(x, y, s)\, ds \quad . \quad \text{ALLORA}$$

$\vec{F} - \nabla\phi$ è ancora una soluzione e la sua terza componente vale $F_3 - \frac{\partial}{\partial z}\phi = 0$

DUNQUE MI RICONDUCO A $\vec{F} = \begin{pmatrix} F_1 \\ F_2 \\ 0 \end{pmatrix}$

IN QUESTA SITUAZIONE

$$rot \begin{pmatrix} F_1 \\ F_2 \\ 0 \end{pmatrix} = \begin{pmatrix} F_{3y} - F_{2z} \\ F_{1z} - F_{3x} \\ F_{2x} - F_{1y} \end{pmatrix} = \begin{pmatrix} -F_{2z} \\ F_{1z} \\ F_{2x} - F_{1y} \end{pmatrix}$$

e quindi la condizione $\text{rot}\, F = \vec{f}$ diventa

$$\left[-\frac{\partial}{\partial z} F_2 = f_1 \;,\; \frac{\partial}{\partial z} F_1 = f_2 \;,\; \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = f_3 \right] (*)$$

Applichiamo questo discorso all'esempio $\vec{f} = \begin{pmatrix} 2x-2y \\ -2x \\ -2z-3 \end{pmatrix}$

$$\frac{\partial}{\partial z} F_2 = -2x+2y \qquad\qquad \frac{\partial}{\partial z} F_1 = -2x$$

$$F_2(x,y,z) = 2(y-x)z + c(x,y) \qquad\qquad F_1 = -2xz + d(x,y)$$

$$\frac{\partial}{\partial x} F_2 - \frac{\partial}{\partial y} F_1 = -2z + \frac{\partial}{\partial x} c(x,y) - \frac{\partial}{\partial y} d(x,y) = -2z-3$$

$$\Longleftrightarrow \quad \frac{\partial}{\partial x} c(x,y) - \frac{\partial}{\partial y} d(x,y) = -3$$

per esempio $c(x,y) = -3x$ $\quad d(x,y) = 0 \quad \Longrightarrow$

$$F_1(x,y,z) = -2xz \;;\; F_2(x,y,z) = 2z(y-x) - 3x$$

QUINDI $\vec{F} = \begin{pmatrix} -2xz \\ 2zy - 2zx - 3x \\ 0 \end{pmatrix}$

(non è quello di ieri — però ai fini degli integrali va bene egualmente!

$$\iint_S \vec{f}\cdot\vec{\nu}\, d\sigma = \int_\gamma \vec{F}\cdot d\vec{s}$$ dove γ descrive la circonferenza

$$= \int_0^{2\pi} \vec{F}(\cos\theta, \sin\theta, 0)\cdot(-\sin\theta, \cos\theta, 0)\, d\theta =$$

$$\int_0^{2\pi} F_2(\cos\theta, \sin\theta, 0)\cos\theta\, d\theta =$$

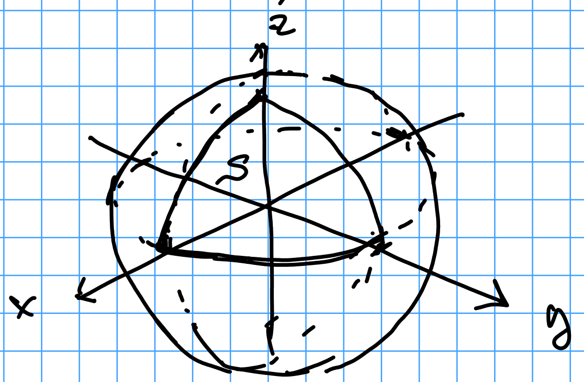
$$\int_0^{2\pi} -3\cos\theta\cos\theta\, d\theta = -3\int_0^{2\pi}\cos^2\theta\, d\theta = -3\pi$$

ALTRI ESEMPI

Calcolare

$$\iint_S \vec{f}\cdot\vec{\nu}\, d\sigma \quad \text{dove} \quad S = \{x^2+y^2+z^2=1,\ x\geq 0, y\geq 0, z\geq 0\}$$

$$\vec{f}(x,y,z) = \begin{pmatrix} x^2 \\ 3xz^2 \\ -2xz \end{pmatrix}$$



($\vec{\nu}$ punta esternamente alla palla piena)

(1) PROVIAMO A CALCOLARLO DIRETTAMENTE

Descrivo S mediante

$$\Gamma(\psi,\theta) = \begin{pmatrix} \cos\theta \, \sin\psi \\ \sin\theta \, \sin\psi \\ \cos\psi \end{pmatrix}$$

$$\Gamma_\psi(\psi,\theta) = \begin{pmatrix} \cos\theta \, \cos\psi \\ \sin\theta \, \cos\psi \\ -\sin\psi \end{pmatrix} \qquad \Gamma_\theta(\psi,\theta) = \sin(\psi) \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$$

$$\Gamma_\psi \otimes \Gamma_\theta = \sin(\psi) \, \det \begin{bmatrix} i & j & k \\ \cos\theta\cos\psi & \sin\theta\cos\psi & -\sin\psi \\ -\sin\theta & \cos\theta & 0 \end{bmatrix}$$

$$\sin(\psi) \begin{pmatrix} \sin\psi \, \cos\theta \\ \sin\psi \, \sin\theta \\ \cos\psi \end{pmatrix} \qquad 0 \le \psi \le \frac{\pi}{2} \quad 0 \le \theta \le \frac{\pi}{2}$$

$$\Phi(\vec{g}, S) = \int_0^{\pi/2}\int_0^{\pi/2} \begin{pmatrix} \cos^2\theta \, \sin^2\psi \\ 3\cos\theta\sin\psi \, \cos^2\psi \\ -2\cos\theta\sin\psi\cos\psi \end{pmatrix} \cdot \begin{pmatrix} \sin\psi \cos\theta \\ \sin\psi \, \sin\theta \\ \cos\psi \end{pmatrix} \sin\psi \, d\psi \, d\theta$$

$$\int_0^{\pi/2}\int_0^{\pi/2} \left(\cos^3\theta \sin^4\psi + 3\cos\theta\sin\theta \, \underline{\cos^2\psi \sin^3\psi} - 2\cos\theta \sin^2\psi\cos^2\psi\right) d\psi \, d\theta$$

$$= \int_0^{\pi/2} \cos^3\theta \, d\theta \int_0^{\pi/2} \sin^4\psi \, d\psi + \frac{3}{2}\int_0^{\pi/2} \sin 2\theta \, d\theta \int_0^{\pi/2} \sin\psi \cos^2\psi (1-\cos^2\psi) \, d\psi$$
$$- 2\int_0^{\pi/2} \cos\theta \, d\theta \int_0^{\pi/2} \sin^2\psi \cos^2\psi \, d\psi$$

$$\int_0^{\pi/2} \cos^3\theta \, d\theta = \int_0^{\pi/2} \cos\theta \,(1-\sin^2\theta)\, d\theta =$$

$$\int_0^{\pi/2} \cos\theta \, d\theta - \int_0^{\pi/2} \cos\theta \, \sin^2\theta \, d\theta$$

$$\overset{\shortparallel}{1} - \left[\frac{\sin^3\theta}{3} \right]_0^{\pi/2} = 1 - \frac{1}{3} = \frac{2}{3}$$

...

CON UN PO' DI PAZIENZA DOVREBBE

VENIRE 1/5

(2) OPPURE cerchiamo $\vec F$ tale che rot $\vec F = \vec g$.

Perché questo sia possibile bisogna che div $\vec g = 0$.

$$\operatorname{div} \vec g = \frac{\partial}{\partial x} x^2 + \frac{\partial}{\partial y} 3xz^2 + \frac{\partial}{\partial z}(-2xz) = 2x + 0 - 2x = 0$$

ok

Cerco $\vec F$ del tipo $\begin{pmatrix} F_1 \\ F_2 \\ 0 \end{pmatrix} \Rightarrow$

$$\frac{\partial}{\partial z} F_2 = -x^2 \; ; \quad \frac{\partial}{\partial z} F_1 = 3xz^2 \; ; \quad \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = -2xz$$

$$F_2 = -x^2 z + c(x,y) \qquad F_1 = xz^3 + d(x,y)$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \cancel{-2xz} + \frac{\partial}{\partial x} c(x,y) - \frac{\partial}{\partial y} d(x,y) = \cancel{-2xz}$$

Posso prendere $c = d = 0 \Rightarrow$

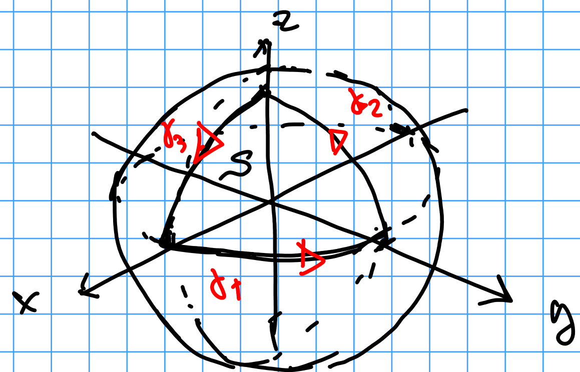
$$F(x,y,z) = \begin{pmatrix} xz^3 \\ -x^2 z \\ 0 \end{pmatrix}$$

APPLICANDO STOKES :

$$\iint_S \vec{g}\cdot\vec{\nu}\, d\sigma = \iint_S \mathrm{rot}\,\vec{F}\cdot\vec{\nu}\, d\sigma = \int_\gamma \vec{F}\cdot d\vec{s}$$

dove γ deve descrivere il bordo di S (coerentemente con ν)

$$\int_\gamma = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3}$$



$\gamma = \gamma_1 + \gamma_2 + \gamma_3$

$\gamma_1(t) = (-\cos(t), \sin(t), 0) \qquad 0 \le t \le \pi/2$

$\gamma_2(t) = (0, -\cos(t), \sin(t)) \qquad 0 \le t \le \pi/2$

$\gamma_3(t) = (\sin(t), 0, \cos(t))$

$$\int_{\gamma_1} \vec{F}\cdot d\vec{s} = \int_0^{\pi/2} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -\sin(t) \\ \cos(t) \\ 0 \end{pmatrix} dt = 0 \qquad (z=0)$$

$$\int_{\gamma_2} \vec{F}\cdot d\vec{s} = \int_0^{\pi/2} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ -\sin(t) \\ \cos(t) \end{pmatrix} dt = 0 \qquad (x=0)$$

$$\int_{\gamma_3} \vec{F}\cdot d\vec{s} = \int_0^{\pi/2} \begin{pmatrix} \sin(t)\cos(t)^3 \\ -\sin^2(t)\cos(t) \\ 0 \end{pmatrix} \begin{pmatrix} \cos(t) \\ 0 \\ -\sin(t) \end{pmatrix} dt = \int_0^{\pi/2} \sin(t)\cos^4(t)\, dt$$

$$- \left[\frac{\cos^5(t)}{5} \right]_0^{\pi/2} = \frac{1}{5}$$

ALTRO ESEMPIO Calcolare

$$\iint_S \vec{f}\cdot\vec{\nu}\, d\sigma$$ dove S è quella di prima (spicchio di sfera)

e $\vec{f}(x,y,z) = \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix}$

VEDIAMO CHE $\mathrm{div}\, \vec{f} = 0$ INFATTI

$$\frac{\partial}{\partial x} yz + \frac{\partial}{\partial y} xz + \frac{\partial}{\partial z} xy = 0 + 0 + 0 = 0$$

CERCO $\vec{F}$ tale che $\mathrm{rot}\, \vec{F} = \vec{f}$; IMPONGO $F_3 = 0 \Rightarrow$

$$\frac{\partial}{\partial z} F_2 = -yz \quad ; \quad \frac{\partial}{\partial z} F_1 = xz \quad ; \quad \frac{\partial}{\partial x} F_2 - \frac{\partial}{\partial y} F_1 = xy$$

$$F_2(x,y,z) = -\frac{yz^2}{2} + c(x,y) \quad ; \quad F_1(x,y,z) = \frac{xz^2}{2} + d(x,y)$$

$$\frac{\partial}{\partial x} F_2 - \frac{\partial}{\partial y} F_1 = \frac{\partial}{\partial x} c(x,y) - \frac{\partial}{\partial y} d(x,y) = xy$$

Per esempio posso prendere $d=0$ e $c(x,y) = \frac{x^2}{2} y$

$$\Rightarrow \quad F(x,y,z) = \frac{1}{2} \begin{pmatrix} xz^2 \\ x^2 y - y z^2 \\ 0 \end{pmatrix}$$

SE VOLESSI TUTTE LE $\vec{F}$ tali che $\operatorname{rot} F = \vec{f}$ avrei

$$\vec{F}(x,y,z) = \frac{1}{2} \begin{pmatrix} xz^2 \\ x^2 y - y z^2 \\ 0 \end{pmatrix} + \nabla \phi$$

Per esempio posso prendere ϕ tale che $\frac{\partial \phi}{\partial y} = \frac{y z^2}{2}$

per esempio $\phi = \frac{y^2 z^2}{4}$. TROVO $\nabla \phi = \begin{pmatrix} 0 \\ \frac{y z^2}{2} \\ \frac{y^2 z}{2} \end{pmatrix}$

$$\vec{F}_1(x,y,z) = \frac{1}{2} \begin{pmatrix} x z^2 \\ y x^2 \\ z y^2 \end{pmatrix}$$

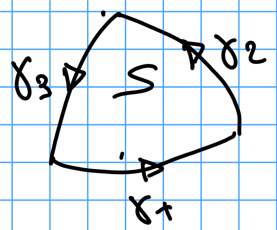
(anche $\vec{F}_1$ ha rotore $\vec{f}$)

Applicando Stokes :

$$\iint_S \vec{g}\cdot\vec{\nu}\, d\sigma = \iint_S \operatorname{rot}\vec{F_1}\cdot\vec{\nu}\, d\sigma =$$

$$\int_{\gamma_1+\gamma_2+\gamma_3} \vec{F_1}\cdot d\vec{s}$$

($\gamma_1 / \gamma_2 / \gamma_3$ quelle di prima:)



$$\int_{\gamma_1} \vec{F_1}\cdot d\vec{s} = \int_0^{\pi/2} \frac{1}{2}\begin{pmatrix} \cos t \cdot 0^2 \\ \sin t \cdot \cos^2 t \\ 0 \cdot \sin^2 t \end{pmatrix}\begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} dt = \frac{1}{2}\int_0^{\pi/2} \sin(t)\cos^3(t)\, dt$$

$$\int_{\gamma_2} \vec{F_1}\cdot d\vec{s} = \int_0^{\pi/2} \frac{1}{2}\begin{pmatrix} 0 \cdot \sin^2 t \\ \cos t \cdot 0 \\ \sin t \ \cos^2 t \end{pmatrix}\begin{pmatrix} 0 \\ -\sin t \\ \cos t \end{pmatrix} dt = \frac{1}{2}\int_0^{\pi/2} \sin(t)\cos^3(t)\, dt$$

$$\int_{\gamma_3} \vec{F_1}\cdot d\vec{s} = \int_0^{\pi/2} \frac{1}{2}\begin{pmatrix} \sin(t)\cos^2(t) \\ 0 \cdot \sin^2(t) \\ \cos(t) \cdot 0^2 \end{pmatrix}\begin{pmatrix} \cos(t) \\ 0 \\ -\sin(t) \end{pmatrix} dt = \frac{1}{2}\int_0^{\pi/2} \sin(t)\cos^3(t)\, dt$$

IL TUTTO $= \dfrac{3}{2}\displaystyle\int_0^{\pi/2} \sin(t)\cos^3(t)\, dt = -\dfrac{3}{2}\left[\dfrac{\cos^4(t)}{4}\right]_0^{\pi/2} = \dfrac{3}{8}$

VERIFICA (integro direttamente $\vec{g}$ su S)

$$T(\psi,\theta) = \begin{pmatrix} \sin\psi \cos\theta \\ \sin\psi \sin\theta \\ \cos\psi \end{pmatrix}$$

(VISTO PRIMA)

$$T_\psi \otimes T_\theta = \sin(\psi) \begin{pmatrix} \sin\psi \cos\theta \\ \sin\psi \sin\theta \\ \cos\psi \end{pmatrix}$$

$0 \leq \theta \leq \frac{\pi}{2}$ $\quad 0 \leq \psi \leq \frac{\pi}{2}$

$$\iint_S \vec{f} \cdot \vec{v} \, d\sigma =$$

$$\int_0^{\pi/2} \int_0^{\pi/2} \begin{pmatrix} \sin\psi \sin\theta \cos\psi \\ \sin\psi \cos\theta \cos\psi \\ \sin^2\psi \cos\theta \sin\theta \end{pmatrix} \begin{pmatrix} \sin\psi \cos\theta \\ \sin\psi \sin\theta \\ \cos\psi \end{pmatrix} \sin\psi \, d\psi \, d\theta =$$

$$3 \int_0^{\pi/2} \int_0^{\pi/2} \sin^3\psi \cos\psi \sin\theta \cos\theta \, d\theta \, d\psi =$$

$$3 \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta \int_0^{\pi/2} \cos\psi \sin^3\psi \, d\psi =$$

$$\frac{3}{2} \left[-\frac{\cos(2\theta)}{2} \right]_0^{\pi/2} \left[\frac{\sin^4\psi}{4} \right]_0^{\pi/2} = \frac{3}{2} \frac{1}{2} (1+1) \frac{1}{4} = \frac{3}{8}$$

TORNA