Analisi Matematica II

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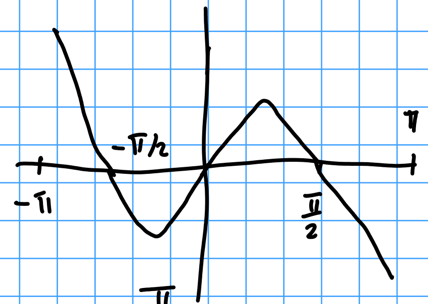
ricevimento: il venerdì alle 11.00 - via Buonarroti 1/c,
oppure su appuntamento (da concordare via email)

• $f(t) = t\cos(t)$ su $[-\pi, \pi]$



$\Rightarrow \omega = 1$

Calcoliamo i coefficienti complessi $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} t\cos(t)\, e^{-int}\, dt$

Dato che $\cos(t) = \frac{e^{it}+e^{-it}}{2}$ prendiamo prima $g(t) = t e^{it}$

e calcoliamo i suoi coefficienti γ_n:

$$\gamma_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\, e^{-int}\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} t\, e^{it} e^{-int}\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} t\, e^{(1-n)it}\, dt$$

NOTIAMO CHE $\gamma_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} t\, dt = \underline{\underline{0}}$; se invece $n \neq 1$

$$\gamma_n \underset{\text{(PER PARTI)}}{=} \frac{1}{2\pi}\left[\frac{t\, e^{(1-n)it}}{(1-n)i}\right]_{-\pi}^{\pi} - \frac{1}{2\pi}\frac{1}{(1-n)i}\underbrace{\int_{-\pi}^{\pi} e^{(1-n)it}\, dt}_{=0} =$$

$e^{i\alpha} + e^{-i\alpha} = 2\cos(\alpha)$

$$\frac{1}{2\cancel{\pi}}\frac{1}{(1-n)i}\left(\cancel{\pi}\, e^{(1-n)i\pi} + \cancel{\pi}\, e^{-(1-n)i\pi}\right) = \frac{1}{2}\frac{i}{n-1}\left(e^{(1-n)i\pi} + e^{-(1-n)i\pi}\right) =$$

$$\frac{i}{n-1}\cos((1-n)\pi) = \frac{i}{n-1}(-1)^{1-n} = -\frac{i}{n-1}(-1)^{-n} = \boxed{-\frac{i}{n-1}(-1)^{n}}$$

DUNQUE $$\gamma_n = \begin{cases} 0 & \text{se } n = 1 \\ -\dfrac{i}{n-1}(-1)^n & \text{se } n \neq 1 \end{cases}$$

Se si considera il secondo pezzo $t\, e^{-it}$ si vede che è $\overline{g(t)}$ e quindi (PER UN FATTO GENERALE) i suoi coefficienti sono

$$\overline{\gamma_{-n}} = \begin{cases} 0 & \text{se } n = -1 \\ \overline{\dfrac{-i}{-n-1}(-1)^{-n}} = \overline{\dfrac{i}{n+1}}(-1)^n = -\dfrac{i}{n+1}(-1)^n & \text{se } n \neq -1 \end{cases}$$

Dato che, come detto sopra $f(t) = \dfrac{g(t) + \overline{g(t)}}{2} \Rightarrow$

$$c_n = \frac{\gamma_n + \overline{\gamma_{-n}}}{2} = \begin{cases} \dfrac{i}{2} & \text{se } n = 1 \\ -\dfrac{(-1)^n i}{2}\left(\dfrac{1}{n+1} + \dfrac{1}{n-1}\right) = -(-1)^n i \dfrac{n}{n^2-1} & \text{se } n \neq \pm 1 \\ -\dfrac{i}{2} & \text{se } n = -1 \end{cases}$$

PASSANDO ALLA FORMA REALE $\Rightarrow$ $a_n = 0$ (perché $\operatorname{Re}(c_n) = 0$)

$$b_n = -2\operatorname{Im}(c_n) = \begin{cases} -1/2 & \text{se } n = 1 \\ (-1)^n \dfrac{2n}{n^2-1} & \text{se } n \neq 1 \end{cases}$$

e quindi

$$t\cos(t) = -\frac{1}{2}\sin(t) + 2\sum_{n=2}^{\infty} \frac{(-1)^n n}{n^2-1}\sin(nt) \quad \left(\begin{array}{l}\text{convergenza puntuale } \forall \\ t \neq \pi + 2k\pi\end{array}\right)$$

SE PER. ES. METTIAMO $t = \frac{\pi}{2}$ allora

$$0 = -\frac{1}{2} + 2\sum_{n=2}^{\infty} (-1)^n \frac{n}{n^2-1} \sin\left(n\frac{\pi}{2}\right) \iff$$

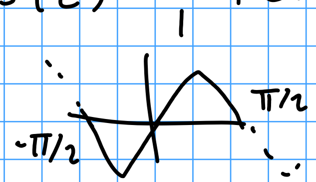
se $n = 2k \Rightarrow \sin\left(n\frac{\pi}{2}\right) = 0$

se $n = 2k+1 \Rightarrow \sin\left(n\frac{\pi}{2}\right) = (-1)^k$

$$\frac{1}{2} = 2\sum_{k=1}^{\infty} \frac{2k+1}{(2k+1)^2-1}(-1)^{2k+1}(-1)^k =$$

$$-2\sum_{k=1}^{\infty} \frac{2k+1}{4k^2+4k}(-1)^k \iff \sum_{k=1}^{\infty} \frac{2k+1}{k(k+1)}(-1)^k = -1$$

CONSIDERIAMO ORA LA STESSA $f(t) = t\cos(t)$, PERÒ SOLO su $[-\pi/2, \pi/2]$ (e quindi π-periodica)



IN QUESTO CASO $\omega = 2$. RIFACENDO I CALCOLI FATTI SOPRA

$$\gamma_m = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} t\, e^{it}\, e^{-i2mt}\, dt = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} t\, e^{(1-2n)it}\, dt =$$

(per parti – ora non ci sono valori di m particolari, dato che $1-2n \neq 0\ \forall n$)

$$\frac{1}{\pi}\left[\frac{t\, e^{(1-2m)it}}{(1-2n)i}\right]_{-\pi/2}^{\pi/2} - \frac{1}{\pi}\frac{1}{(1-2n)i}\int_{-\pi/2}^{\pi/2} e^{(1-2n)it}\, dt =$$

$$\frac{1}{\cancel{\pi}}\frac{i}{2n-1}\left(\frac{\cancel{\pi}}{2}e^{(1-2n)i\frac{\pi}{2}} + \frac{\cancel{\pi}}{2}e^{-(1-2n)i\frac{\pi}{2}}\right) - \frac{1}{\pi}\frac{i}{2n-1}\left[\frac{e^{(1-2n)it}}{(1-2n)i}\right]_{-\pi/2}^{\pi/2} =$$

$$\frac{i}{2n-1}\cos\left((1-2n)\frac{\pi}{2}\right) - \frac{1}{\pi}\left(\frac{i}{2n-1}\right)^2\left(e^{(1-2n)i\frac{\pi}{2}} - e^{-(1-2n)i\frac{\pi}{2}}\right) =$$

$$\frac{i}{2n-1}\underbrace{\cos\left(\frac{\pi}{2}-m\pi\right)}_{= 0} + \frac{1}{\pi}\frac{1}{(2n-1)^2}\, 2i \cdot \underbrace{\sin\left(\frac{\pi}{2}-m\pi\right)}_{= (-1)^m} = \frac{2(-1)^m i}{\pi}\frac{1}{(2n-1)^2}$$

TORNANDO ALLA $f(x) = x\cos(x)$, e ragionando come prima

$$c_m \left(= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} f(t)\, e^{-imt}\, dt\right) = \frac{\gamma_m + \overline{\gamma_{-m}}}{2} = \frac{(-1)^m i}{\pi}\frac{1}{(2n-1)^2} + \frac{(-1)^m(-i)}{\pi}\frac{1}{(2n+1)^2} =$$

$$\frac{(-1)^n i}{\pi}\left(\frac{1}{(2n-1)^2}-\frac{1}{(2n+1)^2}\right)=\frac{(-1)^n i}{\pi}\,\frac{(2n+1)^2-(2n-1)^2}{(2n-1)^2(2n+1)^2}=$$

$$\frac{(-1)^n i}{\pi}\,\frac{\left[(2n+1)-(2n-1)\right]\left[(2n+1)+(2n-1)\right]}{(4n^2-1)^2}=$$

$$\frac{(-1)^n i}{\pi}\,\frac{8n}{(4n^2-1)^2}$$

DUNQUE $a_n=0$ $\forall n$ (perché i c_n sono immaginari puri

$$b_n=-2\,\mathrm{Im}(c_n)=-\frac{8(-1)^n}{\pi}\,\frac{n}{(4n^2-1)^2}$$

cioè
$$t\cos(t)=-\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n n}{(4n^2-1)^2}\sin(2nt)\qquad(*)$$

NOTA CHE $\sum_n |b_n|<+\infty$ $\sum_n n|b_n|<+\infty$, che è legato al fatto che $f(t)$ è C^1 su $\mathbb{R}$.