Analisi Matematica II

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ricevimento: il venerdì alle 11.00 - via Buonarroti 1/c,
oppure su appuntamento (da concordare via email)

$$y' = \frac{\sin(y)}{x^2+y^2} \qquad y(x_0) = x_0$$

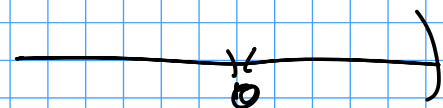
$$\Omega = \mathbb{R}^2 \setminus \{(0,0)\} \qquad F(x,y) = \frac{\sin(y)}{x^2+y^2}$$

$\exists \frac{\partial}{\partial y} F(x,y)$ se $(x,y) \neq (0,0)$ $\Rightarrow$ vale il Th. di

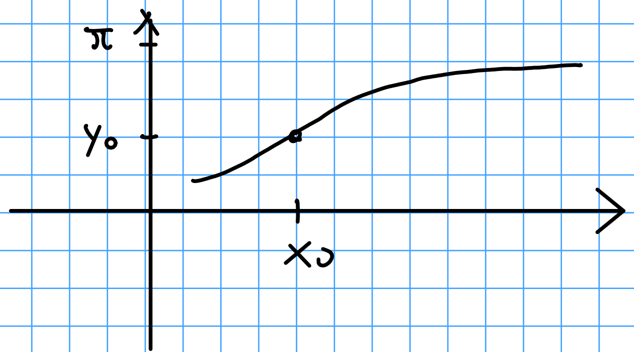
Cauchy, se $(x_0, y_0) \neq (0,0)$.

- se $y_0 = k\pi$ la sol. è costante

(nel caso $x_0 = 0$ devo escludere $(0,0)$)



METTIAMOCI IN $0 < y_0 < \pi$ e $x_0 > 0$



se $]\underline{x}, \bar{x}[$ è l'int. massimale

(1) $\bar{x} = +\infty$, $y(x) < \pi$ $\forall x \in]\underline{x}, \bar{x}[$

$y(x) > 0$ $\forall x \in]\underline{x}, \bar{x}[$ $\quad$ y è crescente su $]\underline{x}, x$

Per $x < x_0$ ci sono due possibilità:

- $\underline{X} = -\infty$
- $\underline{X} = 0$ e $y(x) \to 0$ se $x \to 0^+$

Dato che $y(x)$ è crescente $\Rightarrow \exists\ \bar{y} = \lim_{x \to +\infty} y(x)$

$y_0 < \bar{y} \leq \pi$

PER CAPIRE DOVE È $\bar{y}$ CONSIDERO UN'ALTRA EQ.

$$(\mathrm{I}) \qquad y_1' = \frac{\sin(y_1)}{x^2 + \pi^2} =: F_1(x, y_1)$$

$$(\mathrm{II}) \qquad y_2' = \frac{\sin(y_2)}{x^2 + y_0^2} =: F_2(x, y_2)$$

PER I TEOREMI DI CONFRONTO, DEVE ESSERE

$$y_1(x) \leq y(x) \leq y_2(x) \qquad \forall x \geq x_0$$

(DATO CHE $F_1(x,y) \le F(x,y) \le F_2(x,y)$ se $x \ge x_0$

$y_0 \le y \le \pi$

LE EQ. (I) e (II) sono a variabili separabili

del tipo ☆ $y' = \dfrac{\sin(y)}{x^2+a^2}$ $\quad (a = y_0/\pi)$

POSSIAMO PROVARE A RISOLVERE ☆, con dato $y(x_0) = y_0$

$$\frac{y'(x)}{\sin(y(x))} = \frac{1}{x^2+a^2} \iff \text{(integro tra } x_0 \text{ e } x)$$

$$\int_{x_0}^{x} \frac{y'(x)\,dx}{\sin(y(x))} = \int_{x_0}^{x} \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)$$

$$\parallel$$

$$\int_{y_0}^{y(x)} \frac{ds}{\sin(s)} = \int_{\sin(y_0)}^{\sin(y(x))} \frac{dt}{t\sqrt{1-t^2}} = \int \ldots$$

$t = \sin(s) \qquad s = \arcsin(t) \qquad ds = \dfrac{dt}{\sqrt{1-t^2}}$

MI DICONO CHE

$$\int \frac{ds}{\sin(s)} = \ln\left(\tan\left(\frac{s}{2}\right)\right) \Rightarrow$$

$$\ln\left(\tan\frac{y(x)}{2}\right) - \ln\left(\tan\frac{y_0}{2}\right) = \frac{1}{a}\left(\arctan\left(\frac{x}{a}\right) - \arctan\left(\frac{x_0}{a}\right)\right)$$

$$\tan\left(\frac{y(x)}{2}\right) = \tan\left(\frac{y_0}{2}\right) \cdot e^{\frac{1}{a}\left(\arctan\left(\frac{x}{a}\right) - \arctan\left(\frac{x_0}{a}\right)\right)}$$

$$y(x) = 2\arctan\left[\tan\left(\frac{y_0}{2}\right) \cdot e^{\frac{\arctan\left(\frac{x}{a}\right) - \arctan\left(\frac{x_0}{a}\right)}{a}}\right]$$

(se $x = x_0 \Rightarrow y = y_0$)

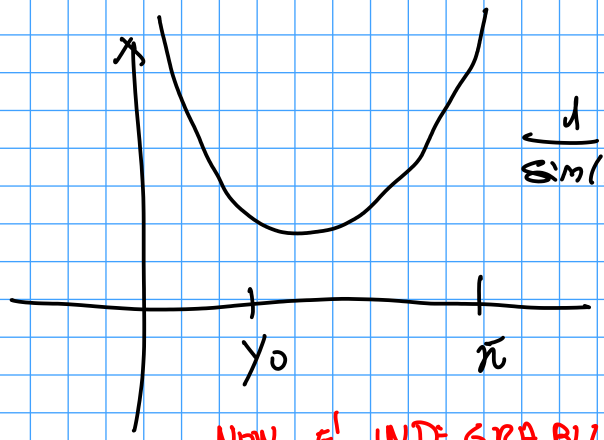
$$\lim_{x \to +\infty} y(x) = 2\arctan\left[\tan\left(\frac{y_0}{2}\right) e^{\frac{1}{a}\left(\frac{\pi}{2} - \arctan\left(\frac{x_0}{a}\right)\right)}\right]$$

$$< 2\arctan\left[\tan\left(\frac{y_0}{2}\right)\right.$$

??

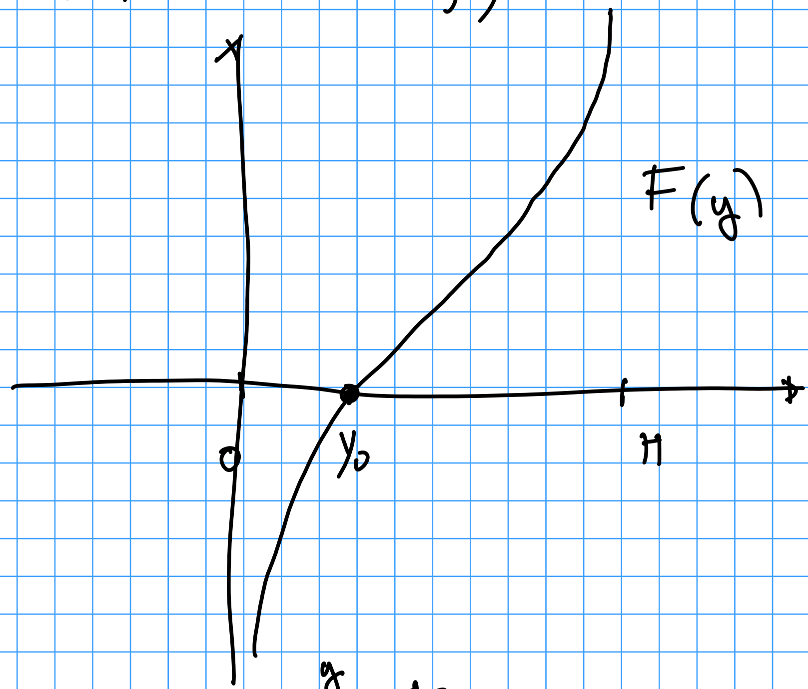
SENZA FARE I CALCOLI SO CHE

$$\int_{y_0}^{y(x)} \frac{ds}{\sin(s)} = \frac{1}{a}\left(\arctan\left(\frac{x}{a}\right) - \arctan\left(\frac{x_0}{a}\right)\right)$$



$\frac{1}{\sin(s)}$

$\Rightarrow$

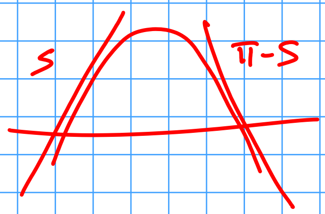


NON E' INTEGRABILE

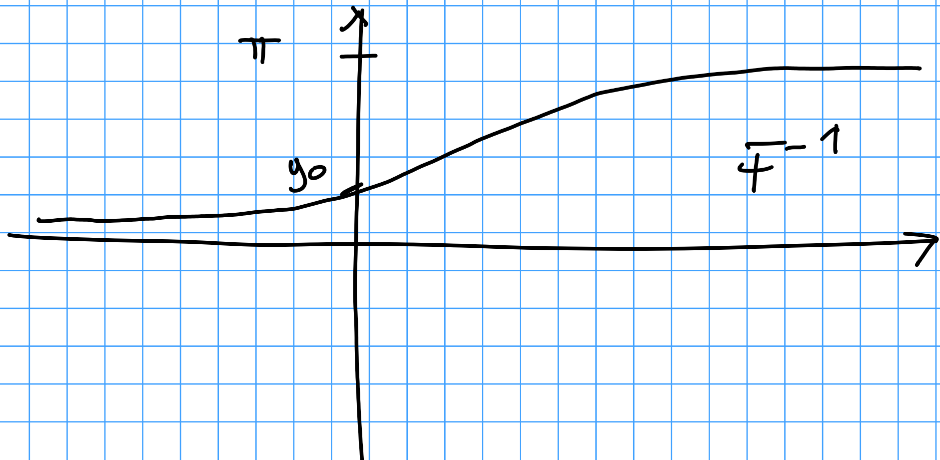
NÉ IN 0 NÉ IN π

$\sin(s) \sim s$ in zero

$\sin(s) \sim \pi - s$ in π



$$F(y) = \int_{y_0}^{y} \frac{ds}{\sin(s)}$$



So che $y(x) = F^{-1}\left(\frac{1}{a}\left(\operatorname{atan}\left(\frac{x}{a}\right) - \operatorname{atan}\left(\frac{x_0}{a}\right)\right)\right)$

$y_e(x)$

I CALCOLI DI PRIMA DICONO CHE

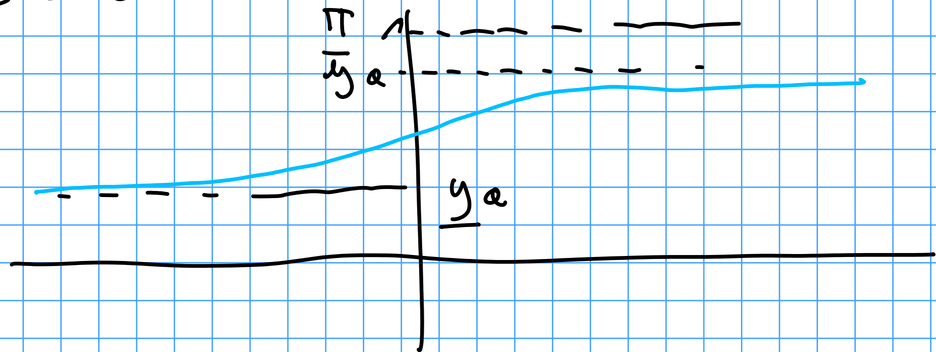
$$F^{-1}(z) = 2 \operatorname{atan}\left(\tan\left(\frac{y_0}{2}\right) e^{z}\right)$$

SI VEDE ALLORA CHE y_e è definito su tutto $\mathbb{R}$

$$\bar{y}_e \quad \lim_{x \to +\infty} y(x) = F^{-1}\left(\frac{1}{a}\left(\frac{\pi}{2} - \operatorname{atan}\left(\frac{x_0}{2}\right)\right)\right) < \lim_{z \to +\infty} F^{-1}(z) = \pi$$

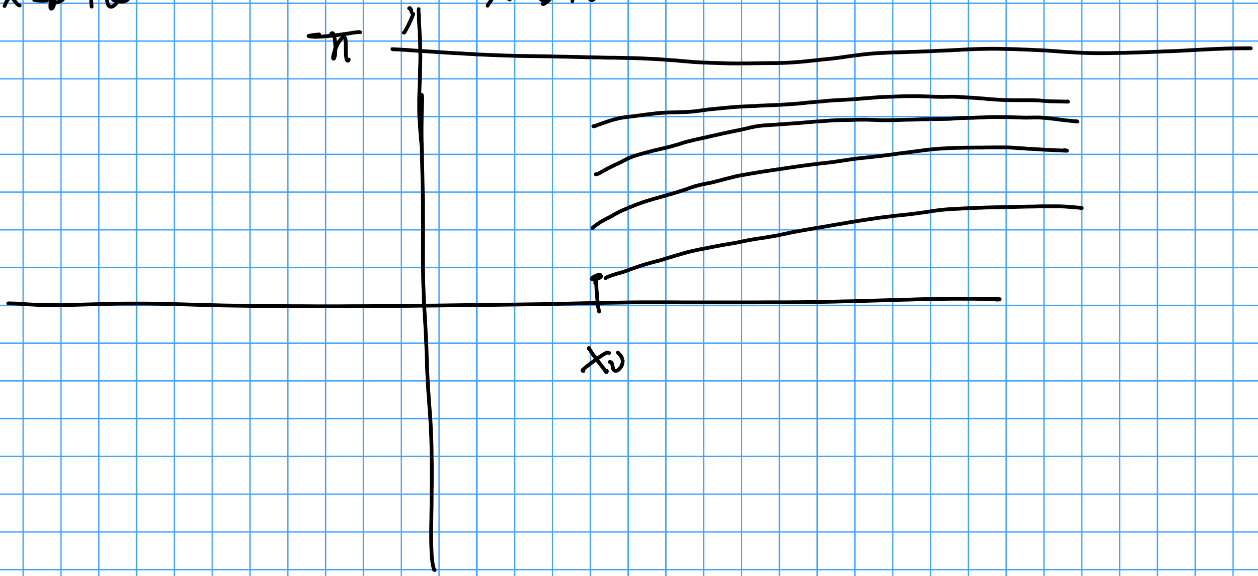
$$\underline{y_e} = \lim_{x \to -\infty} y_e(x) = F^{-1}\left(\frac{1}{a}\left(-\frac{\pi}{2} - \operatorname{atan}\left(\frac{x_0}{a}\right)\right)\right) > \lim_{z \to -\infty} F^{-1}(z) = 0$$

$\Rightarrow$ (grafico di y_e)



TORNANDO ALLA $y(x)$ INIZIALE:

$$\bar{y} := \lim_{x \to +\infty} y(x) \leq \lim_{x \to +\infty} y_{y_0}(x) < \pi$$

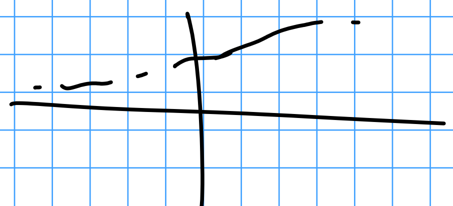


PER VEDERE COSA SUCCEDE "ALL' INDIETRO"

PARTIAMO DA $x_0 = 0$, $y_0 > 0$

QUESTE SOLUZIONI HANNO INTERVALLO MASSIMALE

$= \mathbb{R}$ $(\underline{x} = -\infty, \overline{x} = +\infty)$



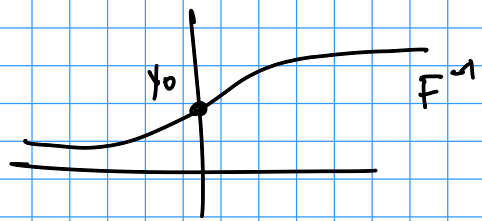
$\forall x \qquad 0 < y(x) < \pi$

$$\frac{\sin(y)}{x^2+y^2} \leq \frac{\sin(y)}{x^2}$$

RISOLVIAMO il caso $a=0$ $\qquad \tilde{y}' = \dfrac{\sin(\tilde{y})}{x^2}$

$$\frac{\tilde{y}'}{\sin(\tilde{y})} = \frac{1}{x^2}$$

$$\Rightarrow \qquad F(\tilde{y}(x)) = \frac{1}{x_0} - \frac{1}{x}$$



$$\tilde{y}(x) = F^{-1}\left(\frac{1}{x_0} - \frac{1}{x}\right)$$

$$\lim_{x \to -\infty} \tilde{y}(x) > 0 \qquad \Rightarrow \qquad \lim_{x \to -\infty} y(x) > 0$$