Analisi Matematica II

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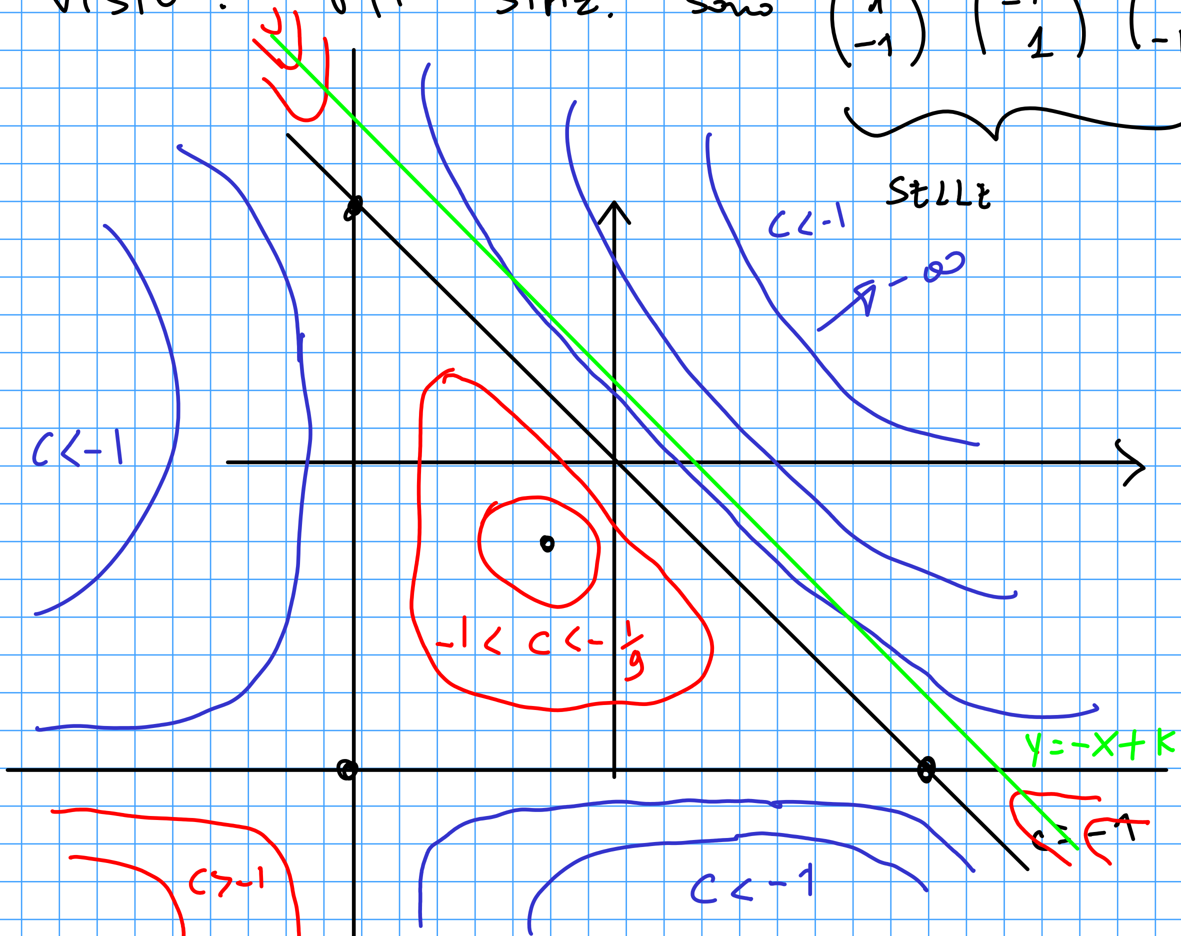
ricevimento: il lunedì dalle 8.30 - via Buonarroti 1/c

Dalle volte scorse: $f(x,y) = x^3 + y^3 - (1+x+y)^3$.

VISTO: PTI STAZ. sono $\underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix}}_{\text{selle}}$ $\underbrace{\begin{pmatrix} -1/3 \\ -1/3 \end{pmatrix}}_{\text{max locale}}$



$f(1,-1) = -1$

$f(-1,1) = -1$

$f(-1,-1) = -1$

$f(-1/3,-1/3) = -\frac{1}{9}$

sulle rette $x=y$ / $x=-y$ trovo: $f(x,x) = 2x^3 - (1+2x)^3 =$
$2x^3 - (1+6x+12x^2+8x^3) = \underline{-1-6x-12x^2-6x^3}$

??

$f(x,-x) = -1$

$f(x,y) = -1$??

$$\boxed{x^3+y^3 = (1+x+y)^3 - 1^3}$$

$$(x+y)(x^2-xy+y^2) = (x+y)\left(1+(1+x+y)+(1+x+y)^2\right)$$

$$\Updownarrow$$

$x+y=0$ oppure

$$\cancel{x^2}-xy+\cancel{y^2} = 1+1+x+y+1+2x+2y+2xy+\cancel{x^2}+\cancel{y^2}$$

$$0 = 3+3xy+3x+3y$$

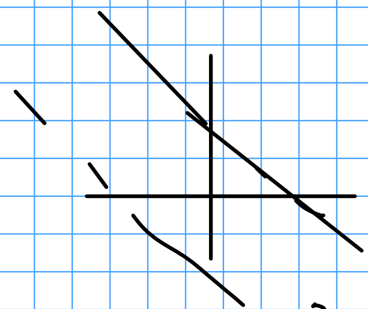
$$x+y+xy+1=0$$

$$y(1+x)+1+x=0 \qquad (y+1)(x+1)=0$$

PER CURIOSITÀ Proviamo a vedere cosa fa

$f(x, k-x)$ al variare di k

se $x \to +\infty / -\infty$



$$f(x, k-x) = x^3+(k-x)^3-(1+x+k-x)^3 =$$

$$x^3+(k^3-3k^2x+3kx^2-x^3)-(1+k)^3$$

$$k^3-3k^2x+3kx^2-(1+k)^3$$

$\nearrow +\infty$ se $|x|\to\infty$, $k>0$

$\searrow -\infty$ se $|x|\to\infty$, $k<0$

ALTRO ESERCIZIO:

$$f(x,y) = \frac{x}{y} + \frac{x}{8} - y = x\left(\frac{1}{8} + \frac{1}{y}\right) - y$$

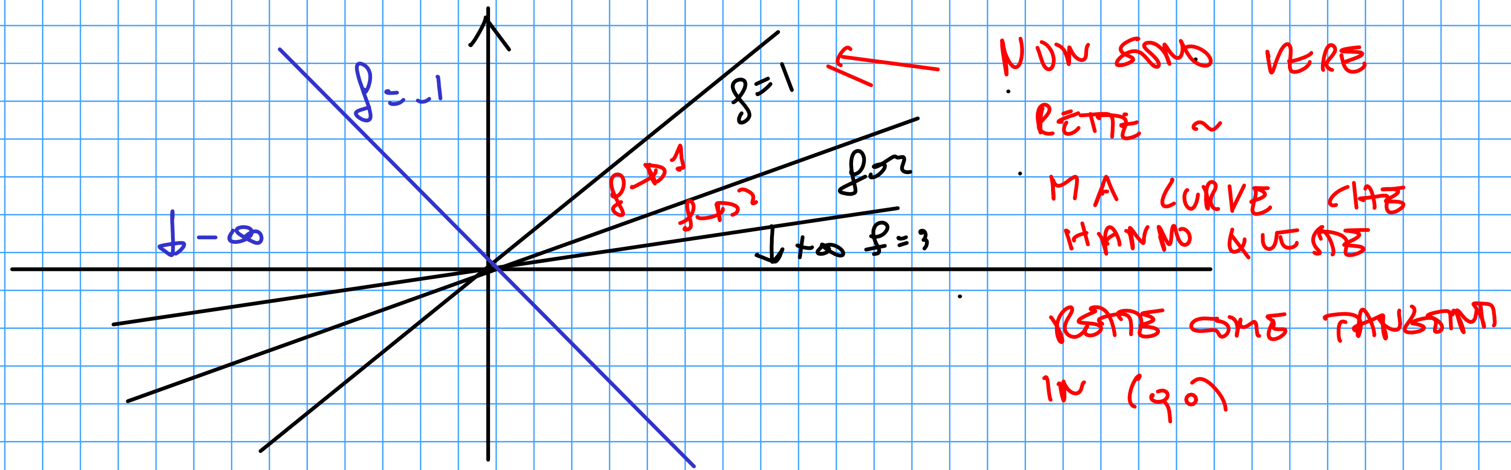
f è definita su $y \neq 0$, cioè su $A = \{(x,y) : y \neq 0\}$
$= \mathbb{R} \setminus \{\text{asse } x\}$

OSSERVIAMO CHE: se $x_0 > 0$

$$\lim_{\substack{(x,y)\to(x_0,0) \\ y>0 \\ (y<0)}} = +\infty \ (-\infty)$$

(se $x_0 < 0$ i limiti si scambiano)

$\left(\forall c \in \mathbb{R} \ \exists U \text{ intorno di } (x_0, 0) : \forall (x,y) \in U, \ y > 0 \Rightarrow f(x,y) > c \right)$



IN $(0,0)$ NON C'È LIMITE $\quad f(mx, x) \to m$

PTI STAZIONARI

$$\frac{\partial f}{\partial x} = \left(\frac{1}{8}+\frac{1}{y}\right) \qquad \frac{\partial f}{\partial y} = \frac{-x}{y^2} - 1$$

pti staz. : $\boxed{y = -8 \qquad x = -64}$

Hessiana:

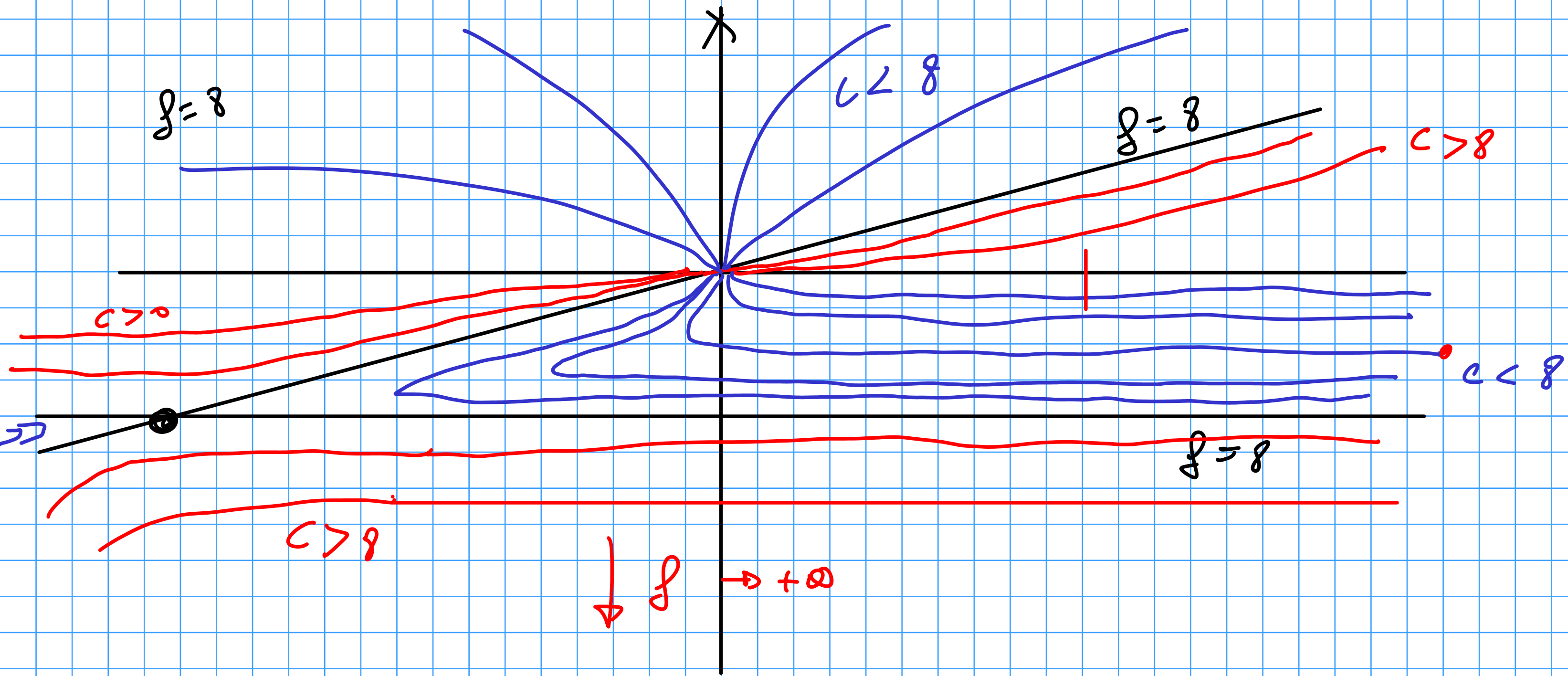
$$\frac{\partial^2 f}{\partial x^2} = 0 \qquad \frac{\partial^2 f}{\partial y \partial x} = -\frac{1}{y^2} \qquad \frac{\partial^2 f}{\partial y^2} = \frac{2x}{y^3}$$

$$H_f(-8,-64) = \begin{pmatrix} 0 & -\frac{1}{64} \\ -\frac{1}{64} & \frac{1}{4} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 0 & -1/16 \\ -\frac{1}{16} & 1 \end{pmatrix}$$

$$\det = -\left(-\frac{1}{16} \cdot -\frac{1}{16}\right) < 0 \quad \Rightarrow \underline{\text{SELLA}}$$

$$f(-64,-8) = -64\left(\frac{1}{8} - \frac{1}{8}\right) + 8 = 8$$

$$x\left(\frac{1}{8}+\frac{1}{y}\right) - y$$



cerchiamo l'insieme $\{(x,y) : f(x,y) = 8\}$ $\Rightarrow$

$$x\left(\frac{1}{8}+\frac{1}{y}\right) = y+8 \iff \frac{x}{8y}(y+8) = y+8$$

$$\left(\frac{x}{8y}-1\right)(y+8) = 0 \iff \frac{x}{8y} = 1 \quad \text{oppure} \quad y=-8$$

ANCORA UNO:

$$f(x,y) = \frac{x}{1+x^2+y^2}$$

DOMINIO $= \mathbb{R}^2$

$$\lim_{\|(x,y)\| \to \infty} f(x,y) = 0 \qquad \left(\text{cioè } \forall \varepsilon > 0 \ \exists R : \text{se } \|(x,y)\| > R \Rightarrow |f(x,y)| < \varepsilon \right)$$

$\Uparrow$

posso in coordinate polari $f(\rho,\theta) = \dfrac{\rho \sin\theta}{1+\rho^2}$

$$|f(\rho,\theta)| \leq \frac{\rho}{1+\rho^2} \to 0 \quad \text{se } \rho \to \infty$$

NOTO CHE $f(x,y) > 0$ se $x > 0$

$f(x,y) < 0$ se $x < 0$

$f(x,y) = 0 \iff x = 0$

PTI STAZ:

$$\frac{\partial f}{\partial x} = \frac{(1+x^2+y^2) - x \cdot 2x}{(1+x^2+y^2)^2} = \frac{1 - x^2 + y^2}{(1+x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{-x\,2y}{(1+x^2+y^2)^2}$$

$$\begin{cases} x^2 - y^2 = 1 \\ xy = 0 \end{cases} \iff \begin{array}{l} x = 0 \text{ e } -y^2 = 1 \quad \text{IMPOSSIBILE} \\ \text{oppure} \\ y = 0 \quad x = \pm 1 \end{array}$$

$\Rightarrow$ 2 PUNTI $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ e $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$

Hessiana
$$\frac{\partial^2 f}{\partial x^2} = \frac{-2x(1+x^2+y^2)^2 - (1-x^2+y^2)2(1+x^2+y^2)2x}{(1+x^2+y^2)^4}$$

$$= \frac{-2x(1+x^2+y^2) - (1-x^2+y^2)4x}{(1+x^2+y^2)^3} =$$

$$= \frac{-2x - 2x^3 - 2xy^2 - 4x + 4x^3 - 4xy^2}{(1+x^2+y^2)^3} =$$

$$= \frac{-6x + 2x^3 - 6xy^2}{(1+x^2+y^2)^3} = \frac{x(-6+2x^2-6y^2)}{(1+x^2+y^2)^3}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{2y(1+x^2+y^2)^2 - (1-x^2+y^2)2(1+x^2+y^2)2y}{(1+x^2+y^2)^4} =$$

$$2y\,\frac{1+x^2+y^2 - 2 + 2x^2 - 2y^2}{(1+x^2+y^2)^3}$$

$$= 2y\,\frac{-1 + 3x^2 - y^2}{(1+x^2+y^2)^3}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{-2x(1+x^2+y^2)^2 + 2xy\,2(1+x^2+y^2)2y}{(1+x^2+y^2)^4} =$$

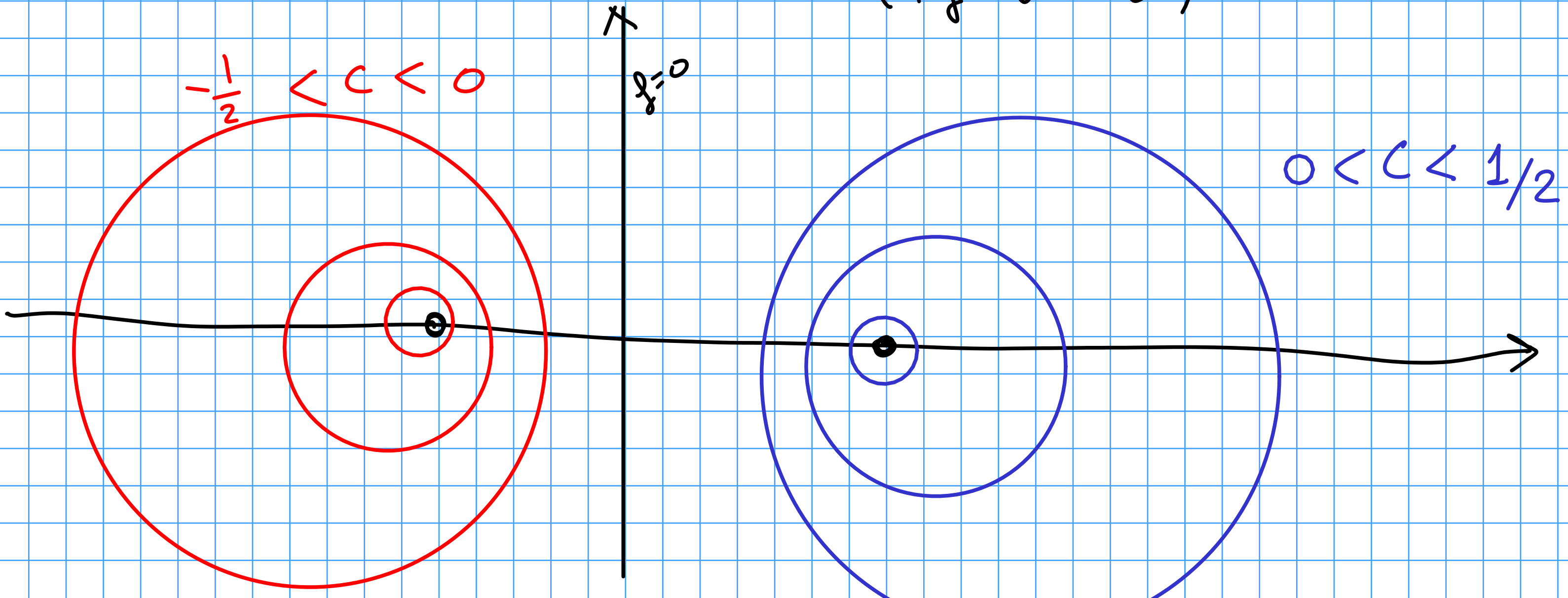
$$2x\,\frac{-(1+x^2+y^2)+4y^2}{(1+x^2+y^2)^3} = \frac{2x(-1-x^2+3y^2)}{(1+x^2+y^2)^3}$$

$$H_f(1,0) = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \Leftarrow \text{MAX} \qquad f(1,0) = \frac{1}{2}$$

$H_f < 0$

$$H_f(-1,0) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \Leftarrow \text{MINIMO} \qquad f(-1,0) = -\frac{1}{2}$$

(H_f è > 0)



NOTA $f(x,y)=c$ viene un cerchio $c\neq 0$ (eventualmente ϕ)

$$\frac{x}{c} = (1+x^2+y^2) \iff$$

$$y^2+\left(x-\frac{1}{2c}\right)^2+1-\frac{1}{4c^2}=0$$

centro in $\left(\frac{1}{2c},0\right)$ raggio $\sqrt{\frac{1}{4c^2}-1}$

SI CAPISCE che

$$\frac{1}{2}=\max_{\mathbb{R}^2} f$$

$$-\frac{1}{2}=\min_{\mathbb{R}^2} f$$