(IERI)

$a_n \to +\infty \implies$

$$\left(1+\frac{1}{a_n}\right)^{a_n} \to e$$

abbiamo usato $\left(r_n := [a_n] \right)$

$$\left(1+\frac{1}{[a_n]}\right)^{[a_n]} \to e$$

$$\Uparrow$$

$$\left(1+\frac{1}{n}\right)^{n} \to e \ ,$$

$a_n = (-1)^n$

$a_{2n} = 1$

(estratte da a_n)

$a_{2n+1} = -1$

limiti diversi $\Rightarrow$ $(-1)^n$ NON HA LIMITE (caso di i e i') Se $|a_n| \to +\infty$

$$\Rightarrow \boxed{\left(1+\frac{1}{a_n}\right)^{a_n} \to e}$$

DIMOSTRATO SE $a_n \to +\infty$ / $a_n \to -\infty$

CASO GENERALE ??

DOVREI DIVIDERE GLI INDICI :

$$\sigma'_n : a_{\sigma'_n} > 0 \quad , \quad \sigma''_n : a_{\sigma''_n} < 0$$

DIMOSTRARE (...)

$$\underbrace{a_{\sigma'_n} \to +\infty}_{(1)} \quad , \quad \underbrace{a_{\sigma''_n} \to -\infty}_{(2)}$$

IN ENTRAMBI I CASI

$$\underbrace{\left(1+\frac{1}{a_{\sigma'_n}}\right)^{a_{\sigma'_n}} \to e}_{(1)} \quad ; \quad \underbrace{\left(1+\frac{1}{a_{\sigma''_n}}\right)^{a_{\sigma''_n}} \to e}_{(2)}$$

$\Rightarrow$ TESI

Se $a_n \to 0$, $a_n \neq 0$

$$(1+a_n)^{\frac{1}{a_n}} \to e$$

<u>Dim.</u> chiamo $a'_n = \frac{1}{a_n} \Rightarrow |a'_n| \to +\infty$

$$\Rightarrow \left(1+\frac{1}{a'_n}\right)^{a'_n} \to e$$

$$\begin{array}{c} \parallel \\ (1+a_n)^{\frac{1}{a_n}} \end{array}$$

Applico il $\ln(\cdot)$ al limite sopra

$$\Rightarrow \ln\left((1+a_n)^{\frac{1}{a_n}}\right) \to \ln(e)$$

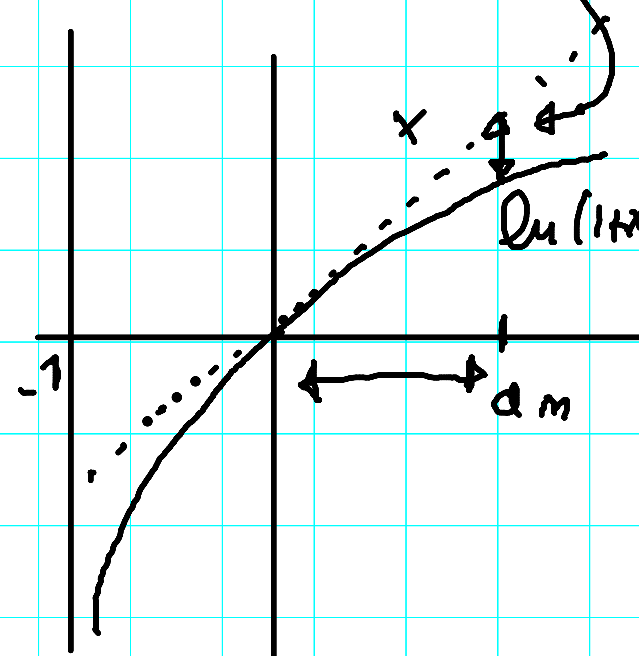
$|\ln(1+a_n) - a_n|$

quindi se $a_n \to 0$

(*) $$\frac{\ln(1+a_n)}{a_n} \to 1$$

$$\Updownarrow$$

$$\ln(1+a_n) = a_n + o(a_n)$$



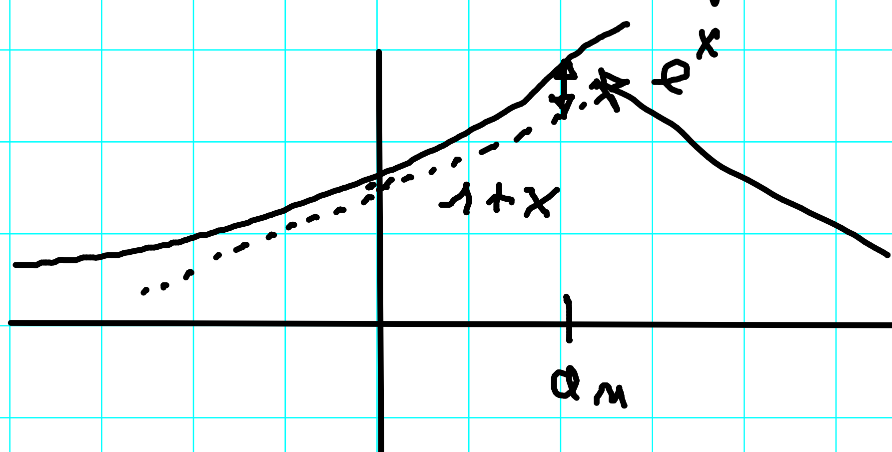
Se $a_n \to 0$

$$\circledast \quad \frac{e^{a_n} - 1}{a_n} \to 1$$

Dim. $b_n = e^{a_n} - 1$; $b_n \to 0$

$$\Rightarrow \frac{\ln(1+b_n)}{b_n} \to 1 \iff \frac{a_n}{e^{a_n} - 1} \to 1$$

posso al reciproco $\Rightarrow$ TESI.



$$e^{a_n} - 1 = a_n + o(a_n) \iff e^{a_n} = 1 + a_n + o(a_n)$$

$a_n \to 0$ $\qquad (\alpha \in \mathbb{R}$ FISSATO$)$

$$\frac{(1+a_n)^\alpha - 1}{a_n} \to \alpha$$

Caso particolare $\alpha = 2$

$$\frac{(1+a_n)^2 - 1}{a_n} = \frac{\cancel{1} + 2\cancel{a_n} + a_n^{\cancel{2}} - \cancel{1}}{\cancel{a_n}} \to 2$$

$$= 2 + a_n \to 2$$

$\alpha = 3$

$$\frac{(1+a_n)^3 - 1}{a_n} = \frac{\cancel{1} + 3a_n + 3a_n^2 + a_n^3 - \cancel{1}}{a_n} \to 3$$

$\alpha = 1/2$

(razionalizzo)

$$\frac{\sqrt{1+a_n} - 1}{a_n} = \frac{1 + a_n - 1}{(\sqrt{1+a_n} + 1)\, a_n} = \frac{1}{\sqrt{1+a_n} + 1} \to \frac{1}{2}$$

$$\underbrace{\frac{(1+a_n)^{\alpha}-1}{a_n}}_{x_n} \longrightarrow \alpha$$

Dim.

$$x_n = \frac{e^{\alpha \ln(1+a_n)}-1}{a_n} =$$

$$\underbrace{\frac{e^{\alpha \ln(1+a_n)}-1}{\alpha \ln(1+a_n)}}_{\downarrow \atop 1} \; \underbrace{\frac{\alpha \ln(1+a_n)}{a_n}}_{\downarrow \atop \alpha \cdot 1}$$

$$\left(b_n = \alpha \ln(1+a_n) \to 0 \Rightarrow \frac{e^{b_n}-1}{b_n} \to 1 \right)$$

$$\lim_{n\to\infty} \sqrt{n^2+1} - n \quad (=0)$$

I° modo (razionalizzando)

$$\frac{\cancel{n^2}+1-\cancel{n^2}}{\sqrt{1+n^2}+n} \to 0$$

II° modo (formule di prima)

$$\sqrt{n^2+1} - n = n\left(\sqrt{1+\frac{1}{n^2}} - 1\right) =$$

$$n\left(\left(1+\frac{1}{n^2}\right)^{\frac{1}{2}} - 1\right) = n\left(\cancel{1} + \frac{1}{2}\frac{1}{n^2} + o\left(\frac{1}{n^2}\right) - \cancel{1}\right)$$

$$\frac{1}{2n} + o\left(\frac{1}{n}\right) \to 0 \quad \left(\text{però ho trovato che} \to 0 \text{ "come } \frac{1}{2n}\text{"}\right)$$

Del II° modo ottengo

$$n\left(\sqrt{n^2+1}-n\right)=n\left(\frac{1}{2n}+o\left(\frac{1}{n}\right)\right)=$$

$$\frac{1}{2}+o(1)\to\frac{1}{2}$$

$$\sqrt{n^2+n}-n=n\left(\left(1+\frac{1}{n}\right)^{\frac{1}{2}}-1\right)=$$

$$n\left(\cancel{1}+\frac{1}{2}\frac{1}{n}+o\left(\frac{1}{n}\right)-\cancel{1}\right)=$$

$$n\left(\frac{1}{2n}+o\left(\frac{1}{n}\right)\right)=\frac{1}{2}+o(1)\to\frac{1}{2}$$

$$\circledast = \sqrt[3]{n^3+n-1} - n \to ?$$

<u>Razionalizzo</u> $\left(A^3 - B^3 = (A-B)(A^2+AB+B^2) \right)$

$A = \sqrt[3]{n^3+n-1} \qquad B = n \qquad \ldots$

$$\circledast = \frac{\cancel{n^3} + n - 1 - \cancel{n^3}}{(n^3+n-1)^{2/3} + n(n^3+n-1)^{1/3} + n^2} =$$

$$\frac{n-1}{n^2\left(\left(1+\frac{1}{n^2}-\frac{1}{n^3}\right)^{2/3} + \left(1+\frac{1}{n^2}-\frac{1}{n^3}\right)^{1/3} + 1 \right)} \to 0$$

INVECE DI RAZIONALIZZARE

$$\sqrt[3]{n^3+n-1} - n =$$

$$n\left(\left(1+\frac{1}{n^2}-\frac{1}{n^3}\right)^{1/3} - 1\right) =$$

$$n\left(\cancel{1}+\frac{1}{3}\left(\frac{1}{n^2}-\frac{1}{n^3}\right)+o\left(\overbrace{\frac{1}{n^2}-\frac{1}{n^3}}^{=O(1/n^2)}\right)-\cancel{1}\right)$$

$$o\left(\frac{1}{n^2}\right)$$

$$n\left(\frac{1}{3}\frac{1}{n^2}+o\left(\frac{1}{n^2}\right)+o\left(\frac{1}{n^2}\right)\right) = n\left(\frac{1}{3n^2}+o\left(\frac{1}{n^2}\right)\right)$$

$$= \boxed{\frac{1}{3n}+o\left(\frac{1}{n}\right)} \to 0$$

NE DEDUCO ANCHE

$$n\left(\sqrt[3]{n^3+n-1}-n\right) = n\left(\frac{1}{3n}+o\left(\frac{1}{n}\right)\right) = \frac{1}{3}+o(1)$$

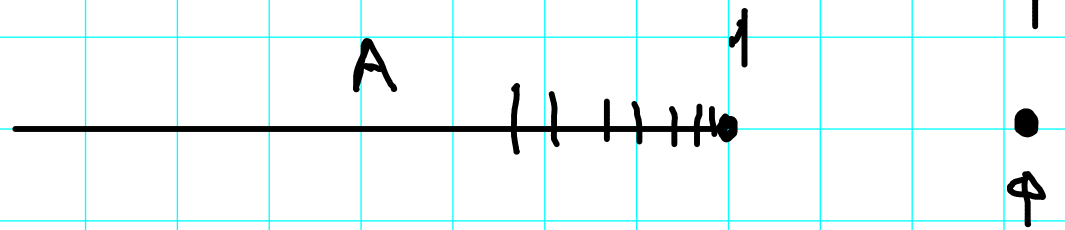
CIOÈ $\displaystyle\lim_{n\to\infty} n\left(\sqrt[3]{n^3+n-1}-n\right) = \frac{1}{3}$

ESERCIZIO

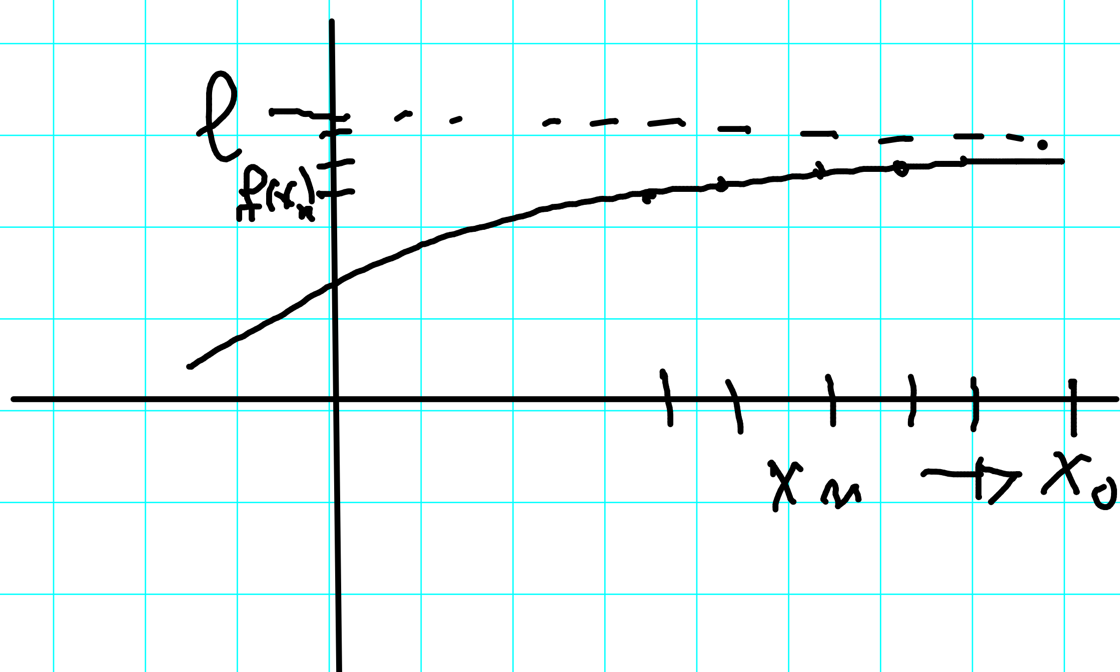
$$\lim_{n\to\infty} \sqrt[3]{n^3-2n^2+1} - n = ??$$

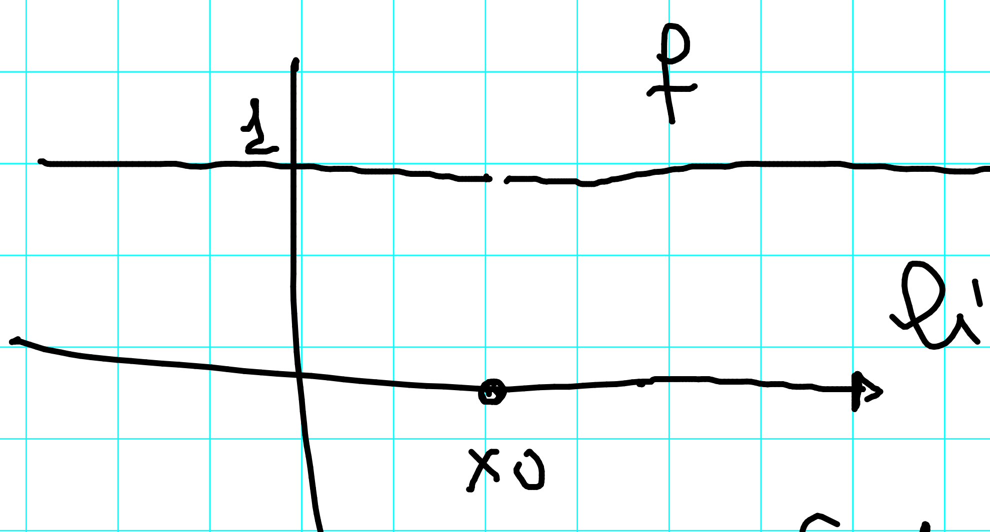
PAUSA

$A = [0,1[$ $\quad 1 = x_0$



$A = [0,1[\cup \{2\}$ $\quad$ ISOLATO

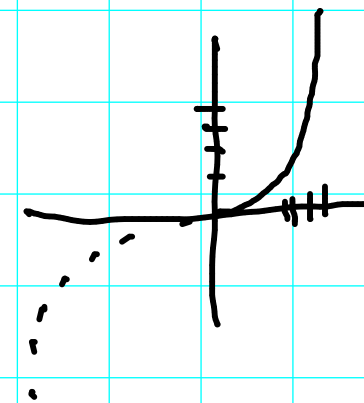




$\lim f(x) = 1$

$$f(x) = \begin{cases} 1 & \text{se } x \neq x_0 \\ 0 & \text{se } x = x_0 \end{cases}$$

$$\lim_{x \to 0^+} x^3 = 0$$



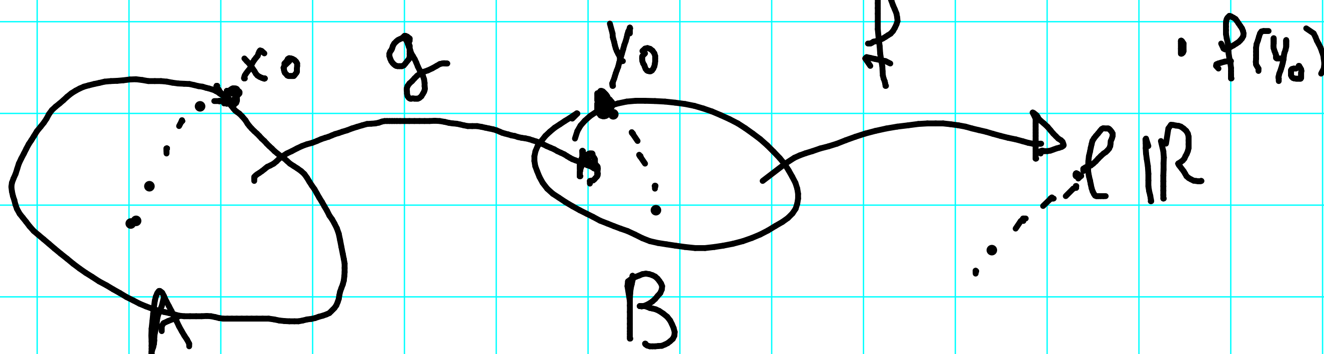
$$\Updownarrow$$

$\forall \{x_n\}$ tale che $x_n > 0$, $x_n \to 0$

$$\Rightarrow \quad x_n^3 \to 0$$

$f : B \to \mathbb{R}$ $\qquad y_0$ DI ACC. PER B

$g : A \to B$ $\qquad x_0$ ACC. PER A



$$g(x) \xrightarrow{x \to x_0} y_0 \qquad f(y) \xrightarrow{y \to y_0} \ell$$

$$\Rightarrow \quad f(g(x)) \xrightarrow{x \to x_0} \ell \qquad \text{NO}$$

INGHIPPO: PUÒ ESSERE

$$g(x_n) = y_0$$

ESEMPIO $\quad f(x) = \begin{cases} 0 & \text{se } x \neq 0 \\ 1 & \text{se } x = 0 \end{cases}$

$f(g(x)) = f(0) = 1 \to 1$

$$g(x) = 0$$

$\lim_{x \to 0} g(x) = 0 \qquad \lim_{y \to 0} f(y) = 0$

Devo mettere un'ipotesi (a scelta)

- $g(x) \neq y_0$ per x vicino a x_0

- $f(y_0) = \ell$

ALLORA LA TESI VALE

FINE (per oggi!!)