$\boxed{A}$

(1) (a) $\boxed{-1}$ (b) $\boxed{-1}$

(2) (a) $\boxed{1/4}$ (b) $\boxed{+\infty}$

(3) $\boxed{1/8}$

(4) (a) $\boxed{3}$ (b) $\boxed{-\infty}$

(6)

~~AC~~	C	NC

AC	C	~~NC~~

(7) $\boxed{\alpha < 1}$

(8) $\boxed{\dfrac{(3+2x+e^x)e^x}{4}}$

(9) $\boxed{\dfrac{6-7\ln(2)}{36}}$

$\boxed{B}$

(1) (a) $\boxed{-2}$ (b) $\boxed{-4}$

(2) (a) $\boxed{\frac{1}{6}}$ (b) $\boxed{+\infty}$

(3) $\boxed{\frac{1}{7}}$

(4) (a) $\boxed{5/4}$ (b) $\boxed{-\infty}$

(6)

~~AC~~	C	NC

AC	C	~~NC~~

(7) $\boxed{\alpha < 1}$

(8) $\boxed{\dfrac{(3+2x+e^{2x})e^{x}}{4}}$

(9) $\boxed{\dfrac{6-7\ln(2)}{18}}$

$\boxed{C}$

(1) (a) $\boxed{-3}$ (b) $\boxed{-9}$

(2) (a) $\boxed{\frac{1}{8}}$ (b) $\boxed{+\infty}$

(3) $\boxed{\frac{1}{6}}$

(4) (a) $\boxed{-\infty}$ (b) $\boxed{2/3}$

(6)

AC	C	~~NC~~

~~AC~~	C	NC

(7) $\boxed{\alpha < 1}$

(8) $\boxed{\dfrac{(-3+2x+3e^{-x})e^{-x}}{4}}$

(9) $\boxed{\dfrac{6-7\ln(2)}{72}}$

D

(1) (a) -4 (b) -16

(2) (a) $\frac{1}{10}$ (b) $+\infty$

(3) $\frac{1}{5}$

(4) (a) $-\infty$ (b) $3/8$

(6)

AC	C	~~NC~~

~~AC~~	C	NC

(7) $\alpha < 1$

(8) $\dfrac{e^{-x}(-3+2x+3e^{-2x})}{4}$

(9) $\dfrac{6-7\ln(2)}{54}$

(1) Usando Taylor si vede che

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + o(x^2), \qquad \cos(ax) = 1 - \frac{a^2x^2}{2} + \frac{a^4x^4}{24} + o(x^4)$$

da cui

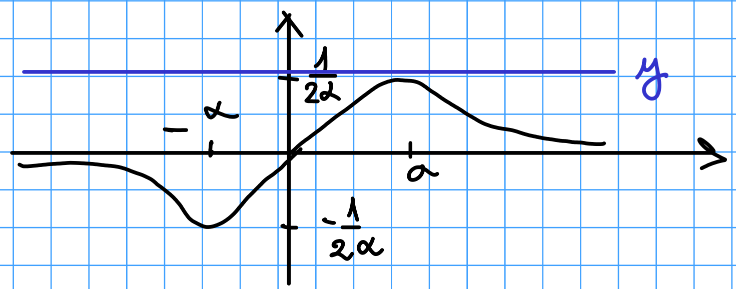
$$f(x) = \sqrt{\cos(ax)} = \sqrt{1 - \frac{a^2x^2}{2} + \frac{a^4x^4}{24} + o(x^4)} = 1 + \frac{1}{2}\left(-\frac{a^2x^2}{2} + \frac{a^4x^4}{24} + o(x^4)\right)$$

$$- \frac{1}{8}\left(\left(-\frac{a^2x^2}{2} + o(x^3)\right)^2\right) + o\left(\left(o(x)\right)^4\right) =$$

$$1 - \frac{1}{4}a^2x^2 + \frac{1}{48}a^4x^4 - \frac{1}{32}a^4x^4 + o(x^4) =$$

$$1 - \underbrace{\frac{a^2}{4}}_{\frac{f''(0)}{2}} x^2 - \underbrace{\frac{a^4}{96}}_{\frac{f^{IV}(0)}{24}} x^4 + o(x^4) \Rightarrow f''(0) = \boxed{-\frac{a^2}{2}} \qquad f^{IV}(0) = \boxed{-\frac{a^4}{4}}$$

(2) Se $f(x) := \dfrac{x}{\alpha^2 + x^2}$ allora f ha il grafico seguente



Ne segue che

$$A = \{y : f(x) < y \ \forall x\} = \left]\frac{1}{2\alpha}, +\infty\right[\Rightarrow$$

$$\inf A = \boxed{\frac{1}{2\alpha}}, \qquad \sup A = \boxed{+\infty}$$

(3) Se $f(x) = 1 + \alpha x + e^{\beta y}$ è chiaro che f è crescente $\Rightarrow$ f invertibile e che $f(0) = 2$. Dunque $(f^{-1})(2) = 0$

e per la formula sulla derivata di f^{-1}:

$$(f^{-1})'(2) = \frac{1}{f'(0)} = \boxed{\frac{1}{\alpha+\beta}}$$

(4) • $$\lim_{n\to\infty} \frac{1+\alpha n^2}{\beta n^2 \sqrt[n]{n+1} + \gamma n \ln(n)} = \lim_{n\to\infty} \frac{\alpha n^2}{\beta n^2} \frac{\frac{1}{\alpha n^2}+1}{\sqrt[n]{n+1} + \frac{\gamma}{\beta}\frac{\ln(n)}{n}} = \boxed{\frac{\alpha}{\beta}}$$

• $$\lim_{n\to\infty} \frac{\alpha n! - n^n}{\beta n! + \gamma^n} = \lim_{n\to\infty} \frac{n^n}{n!} \frac{\alpha\frac{n!}{n^n} - 1}{\beta + \frac{\gamma^n}{n!}} = +\infty \cdot (-1) = \boxed{-\infty}$$

(5) $$\lim_{x\to 0} \frac{e^{2x}\sqrt[3]{1-6x} - \cos(2x)}{1-\cos(x)}.$$ Usiamo Taylor

$$\sqrt[3]{1+y} = 1 + \binom{1/3}{1} y + \binom{1/3}{2} y^2 + o(y^2) = 1 + \frac{y}{3} - \frac{y^2}{9} + o(y^2)$$

$$\Rightarrow \sqrt[3]{1-6x} = 1 - 2x - 4x^2 + o(x^2)$$

$$e^{2x} = 1 + 2x + \frac{(2x)^2}{2} + o(x^2) = 1 + 2x + 2x^2 + o(x^2)$$

NE SEGUE $e^{2x}\sqrt{1-6x} = (1+2x+2x^2+o(x^2))(1-2x-4x^2+o(x^2)) =$

$1 - \cancel{2x} - 4x^2 + o(x^2) + \cancel{2x} - 4x^2 + o(x^2) + 2x^2 + o(x^2) + o(x^2) =$

$1 - 6x^2 + o(x^2)$.

INFINE $\cos(x) = 1 - \frac{x^2}{2} + o(x^2)$ e $\cos(2x) = 1 - 2x^2 + o(x^2)$

$$\Rightarrow \frac{e^{2x}\sqrt{1-6x} - \cos(2x)}{1-\cos(x)} = \frac{1-6x^2+o(x^2) - 1 + 2x^2 + o(x^2)}{1 - 1 + \frac{x^2}{2} + o(x^2)} = \frac{-4x^2+o(x^2)}{\frac{x^2}{2}+o(x^2)} \to \boxed{-8}$$

(6)

- $\sum \underbrace{(-1)^n \frac{n - \sin(n)}{\alpha n + 1}}_{a_n}$ $\boxed{\text{NON CONVERGE}}$, perché $|a_n| = \frac{n}{n} \frac{1 - \frac{\sin(n)}{n}}{\alpha + 1/n} \to \frac{1}{\alpha} \neq 0$

- $\sum \underbrace{(-1)^n \frac{\alpha n^2 - \beta n!}{\gamma n^n - \delta n!}}_{a_n}$ $\boxed{\text{CONVERGE ASSOLUTAMENTE}}$

perché $|a_n| = \frac{n!}{n^n} \frac{\beta + o(1)}{\gamma + o(1)} \simeq \frac{\beta}{\gamma} \frac{n!}{n^n}$ e la serie $\sum_n \underbrace{\frac{n!}{n^n}}_{b_n}$

converge per il criterio del rapporto:

$$\frac{b_{n+1}}{b_n} = \frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!} = \left(\frac{n}{n+1}\right)^n \to \frac{1}{e} < 1$$

(7) $f(x) = \dfrac{x^2 e^{-x}}{\sqrt{|x^2-1|}\,|4-x^2|^\alpha}$ su $[0,+\infty[$ ha due singolarità in $x=1$ e $x=2$

IN $x=1$ $\quad f(x) \simeq \dfrac{e^{-1}}{2^\alpha \sqrt{|x-1|}\cdot 2}$ che è integrabile qualunque sia α

IN $x=2$ $\quad f(x) \simeq \dfrac{4e^{-2}}{\sqrt{3}\,|2-x|^\alpha 4} \simeq \dfrac{c}{|2-x|^\alpha}$ che è integrabile per $\alpha<1$

A $+\infty$ $\quad f(x) \simeq \dfrac{x^2}{x\, x^{2\alpha}} e^{-x}$ che (a causa di e^{-x}) è integrabile $\forall \alpha$

(8) VEDIAMO IL COMPITO DELLA I^a FILA (GLI ALTRI SI FANNO IN MODO ANALOGO)

$y'' - 5y' + 6y = x e^x$ $\qquad P(z) = z^2 - 5z + 6$, radici $2, 3$

$\Rightarrow$ sol. dell'omogenea $\quad y_0(x) = \lambda e^{2x} + \mu e^{3x}$

Posso cercare una sol. particolare del tipo $\bar{y}(x) = (ax+b)e^x \Rightarrow$ trovo

$\bar{y}'(x) = (ax+b+a)e^x \qquad \bar{y}''(x) = (ax+b+2a)e^x \quad \Rightarrow$

$\bar{y}'' - 5\bar{y}' + 6\bar{y} = (ax+b+2a-5ax-5b-5a+6ax+6b)e^x = (2ax-3a+2b)e^x$

da cui $a = \frac{1}{2}$ e $b = \frac{3}{4}$, cioè $\quad \bar{y}(x) = \dfrac{2x+3}{4} e^x$

Dunque $y(x) = \lambda e^{2x} + \mu e^{3x} + \frac{2x+3}{4} e^{x} \Rightarrow y(0) = \lambda + \mu + \frac{3}{4}$

$y'(x) = 2\lambda e^{2x} + 3\mu e^{3x} + \left(\frac{2x+3}{4} + \frac{1}{2}\right) e^{x} \Rightarrow y'(0) = 2\lambda + 3\mu + \frac{5}{4}$

e quindi, imponendo le cond. iniziali: $\begin{cases} \lambda + \mu = \frac{1}{4} \\ 2\lambda + 3\mu = \frac{1}{2} \end{cases} \Leftrightarrow \begin{cases} \lambda = 1/4 \\ \mu = 0 \end{cases}$

(g) $\displaystyle\int_0^{+\infty} \frac{dx}{e^{ax}(e^{2ax} + 4e^{x} + 3}$

pongo $e^{ax} = y \Leftrightarrow a e^{ax} dx = dy$

$\Leftrightarrow dx = \frac{dy}{ay}$

$$= \frac{1}{a} \int_1^{+\infty} \frac{dy}{y^2(y^2 + 4y + 3)}$$

$y^2 + 4y + 3 = 0 \Leftrightarrow y = -2 \pm \sqrt{4-3} = \begin{cases} -3 \\ -1 \end{cases}$

$$= \frac{1}{a} \int_1^{+\infty} \frac{dy}{y^2(y+1)(y+3)} = \frac{1}{a} \int_1^{+\infty} \left[\frac{1}{3} \frac{1}{y^2} + \frac{1}{18}\left(-\frac{8}{y} + \frac{9}{y+1} - \frac{1}{y+3}\right)\right] dy$$

(DATO CHE

$$\frac{A}{y} + \frac{B}{y^2} + \frac{C}{y+1} + \frac{D}{y+3} = \frac{A(y^3 + 4y^2 + 3y) + B(y^2 + 4y + 3) + C(y^3 + 3y^2) + D(y^3 + y^2)}{y^2(y^2 + 4y + 3)}$$

$$\Rightarrow \begin{cases} A + C + D = 0 \\ 4A + B + 3C + D = 0 \\ 3A + 4B = 0 \\ 3B = 1 \end{cases} \quad \begin{cases} B = 1/3 \quad A = -\frac{4}{9} \\ C + D = \frac{4}{9} \\ 3C + D = \frac{13}{9} \end{cases} \quad \begin{cases} A = -\frac{4}{9} \quad B = \frac{1}{3} \\ C = \frac{1}{2} \quad D = -\frac{1}{18} \end{cases}$$)

$$= \frac{1}{a}\left[-\frac{1}{3y} + \frac{1}{18} \ln \left| \frac{(y+1)^9}{y^8(y+3)} \right| \right]_1^{+\infty} = \frac{1}{a}\left(\frac{1}{3} - \frac{1}{18} \ln \frac{2^9}{4} \right) = \frac{1}{a}\left(\frac{1}{3} - \frac{7}{18} \ln (2) \right)$$

10)

$$y' = \frac{2xy}{x^2-9} - 2x^2 \quad \text{per} \quad -3 < x < 3$$

$$A(x) = \int_0^x \frac{2t}{t^2-9}\,dt = \int_0^x \left(\frac{1}{t-3} + \frac{1}{t+3} \right) dt = \ln \left| \frac{x^2-9}{9} \right| = \ln \frac{9-x^2}{9}$$

dato che siamo in $-3 < x < 3$. Allora

$$y(x) = \frac{9-x^2}{9} \left\{ y(0) - 2\int_0^x \frac{9t^2}{9-t^2}\,dt \right\} = (9-x^2)\left\{ \frac{y_0}{9} + 2\int_0^x \frac{t^2}{t^2-9}\,dt \right\} =$$

$$(9-x)^2 \left\{ \frac{y_0}{9} + 2\int_0^x \left(1 + \frac{9}{t^2-9} \right) dt \right\} = (9-x^2)\left\{ \frac{y_0}{9} + 2x + \int_0^x \left(\frac{3}{t-3} - \frac{3}{t+3} \right) dt \right\}$$

$$= (9-x^2)\left\{ \frac{y_0}{9} + 2x + 3\ln\left| \frac{x-3}{x+3} \right| \right\} =$$

$$(9-x^2)\left\{ \frac{y_0}{9} + 2x + 3\ln\left(\frac{3-x}{x+3} \right) \right\}$$

LIMITI

$$\lim_{x \to -3^+} y(x) = 0^+ \left\{ \frac{y_0}{9} - 6 + 3\ln\left(\frac{1}{0^+} \right) \right\} = 0^+ \cdot +\infty$$

DATO CHE IL LOGARITMO "PERDE" VIENE $\boxed{0^+}$

$$\lim_{x \to 3^-} y(x) = 0^+ \left\{ \frac{y_0}{9} + 6 + 3 \ln\left(0^+\right) \right\} = 0^+ \cdot (-\infty)$$

DATO CHE IL LOGARITMO "PERDE" VIENE

$$\boxed{0^-}$$

MONOTONIA CONSIDERIAMO $F(x,y) = \frac{2xy}{x^2-9} - 2x^2$ $(-3 < x < 3)$

di modo che l'equazione diventa $y' = F(x,y)$. Si ha $F(x,y) > 0 \iff$

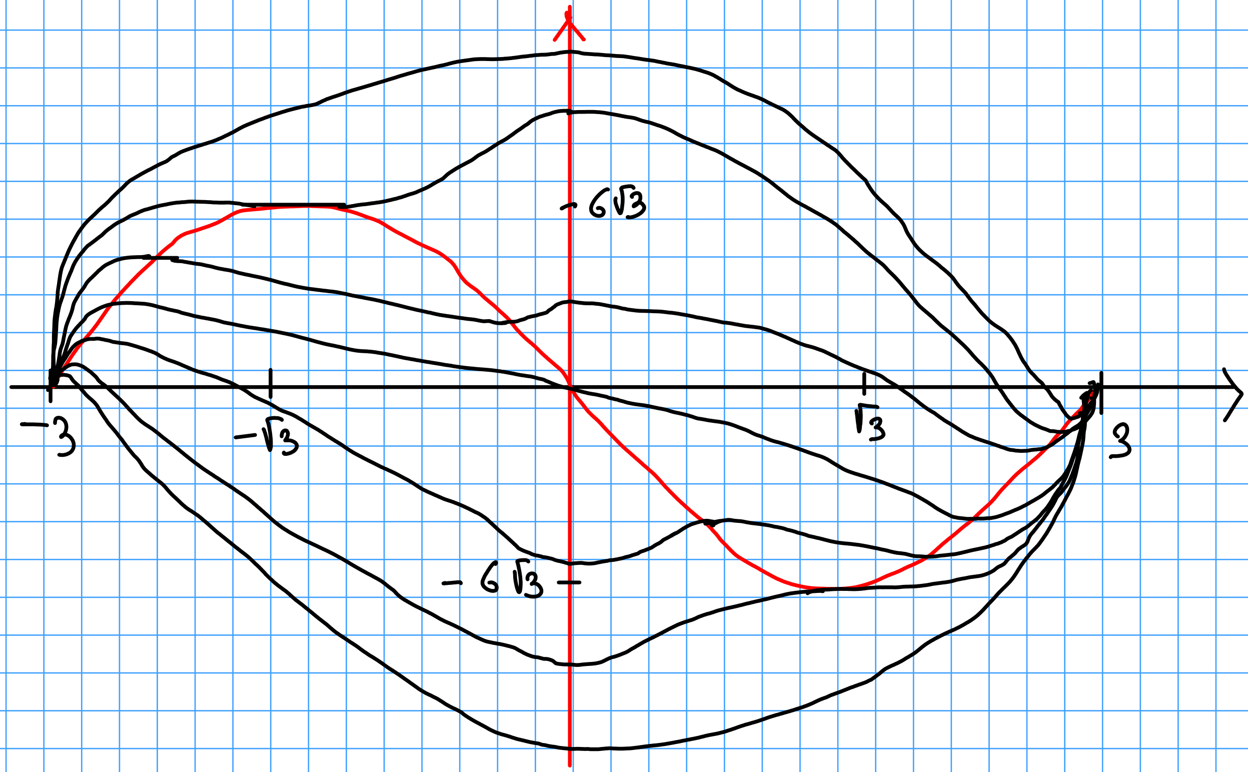
$$y \begin{cases} < x(x^2-9) & \text{se } x > 0 \\ > x(x^2-9) & \text{se } x < 0 \end{cases}$$

(noto che $x^2 - 9 < 0$ in $]-3,3[$)

Prendiamo $g(x) := x(x^2-9)$ e tracciamo il grafico di g (curva rossa). Usando le proprietà delle soluzioni possiamo disegnare le $y(x)$ come nel disegno sotto. In effetti $g'(0) = 0 \iff$

$3x^2 - 9 = 0 \iff x = \pm\sqrt{3}$ e $g(\pm\sqrt{3}) = \mp 6\sqrt{3}$

DAI GRAFICI SI RICAVA CHE $y(x) = 0$ NON HA MAI PIÙ DI UNA SOLUZIONE


6√3
−6√3
−3
−√3
√3
3