1. Let us consider the functional

$$F(u) = \int_0^1 \left(\dot u^2 + \dot u + x^3 u\right)\, dx.$$

(a) Discuss the minimum problem for $F(u)$ subject to the conditions $u(0) + u(1) = 3$.

(b) Discuss the minimum problem for $F(u)$ subject to the conditions $u(0) - u(1) = 3$.

(a) In a standard way we find that

$$\int 2\dot u \dot v + \dot v + x^3 v = 0 \qquad \forall v \in C^1([0,1]) \text{ with } v(0)+v(1) = 0$$

Integrating by parts we obtain that

$$\int [-(2\dot u + 1)' v + x^3 v]\, dx + [(2\dot u + 1) v]_0^1 = 0$$

Limiting ourselves to $v \in C^1_c((0,1))$ we obtain that $2\ddot u = x^3$, and then $(2\dot u(1)+1)v(1) - (2\dot u(0)+1)\underbrace{v(0)}_{=-v(1)} = 0$, which implies $\dot u(1) + \dot u(0) + 1 = 0$.

In conclusion we obtain

$$\begin{cases} 2\ddot u = x^3 \\ u(0) + u(1) = 3 \\ \dot u(0) + \dot u(1) = -1 \end{cases} \rightsquigarrow u(x) = \frac{x^5}{40} + a + bx \rightsquigarrow \boxed{u(x) = \frac{x^5}{40} + \frac{283}{160} - \frac{9}{16} x}$$

This is the unique min. point because for every $v \in C^1([0,1])$ with $v(0) + v(1) = 0$ it turns out that

$$F(u+v) = F(u) + \int_0^1 \dot v^2 \geq F(u)$$

with equality if and only if $\dot v \equiv 0$, namely v constant, and the constant is 0 because $v(0)+v(1) = 0$.

(b) $\boxed{\inf = -\infty}$ and a possible minimizing sequence is

$$u_n(x) = -3x - n.$$

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2. Discuss existence, uniqueness and regularity of the solution to the boundary value problem

$$u'' = u^3 + \sin u + |x-1|^{3/2}, \qquad u(0) = u(2) = 5.$$

Let us consider the Lagrangian

$$L(x,s,p) = \frac{1}{2}\dot{u}^2 + \frac{1}{4}u^4 - \cos u + |x-1|^{3/2} u$$

and the pbm $\min \{ \underbrace{\int_0^2 L(x,u,\dot{u})\,dx}_{F(u)} : u \in H^1((0,2)) + \text{DBCs} \}$

- Existence follows from the direct method. The key point is the estimate

$$F(u) \geq \int_0^1 \frac{1}{2}\dot{u}^2 - 1 - |u|$$

Now we can estimate $|u(x)|$ by means of $\|\dot{u}\|_{L^2}$, and conclude in the usual way.

- Uniqueness follows in the usual way from two facts:
  → every solution to the boundary value pbm is a min. point (due to the convexity of the Lagrangian wrt (s,p))
  → the min. point is unique due to the strict conv. of $L(x,s,p)$ wrt (s,p). Here we need that $s^3 + \sin s$ is strictly increasing. (this has to be verified)

- Regularity. The solution lies in $C^{3,1/2}$ and nothing more. Indeed, computing the first variation we discover that

$$(\dot{u})' = u^3 + \sin u + |x-1|^{3/2}$$

↑ weak derivative

Now the classical bootstrap starts

$u \in H^1 \Rightarrow u \in C^0 \Rightarrow \text{RHS} \in C^0 \Rightarrow \dot{u} \in C^1 \Rightarrow u \in C^2$

$\Rightarrow$ ELE holds true in the classical sense

$\Rightarrow \text{RHS} \in C^{1,1/2}$ (because of $|x-1|^{3/2}$) $\Rightarrow u \in C^{3,1/2}$.

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3. Let us set, for every $\lambda > 0$,

$$I(\lambda) := \inf\left\{\int_0^4 \left(\dot{u}^4 + \dot{u}^2 - \lambda\sin(\dot{u}^2) + xu^2\right) dx : u \in C^1([0,4]),\ u(0) = u(4) = 0\right\}.$$

(a) Determine for which values of λ it turns out that $I(\lambda)$ is a real number.

(b) Determine for which values of λ it turns out that $I(\lambda)$ is actually a minimum.

(c) Determine the limit of $I(\lambda)$ as $\lambda \to +\infty$.

(a) $I(\lambda) \in \mathbb{R}$ $\boxed{\forall \lambda > 0}$. Indeed the Lagrangian is bounded from below (by $-\lambda$).

(b) $I(\lambda)$ is a minimum $\boxed{\forall \lambda \in (0,1]}$. This follows from the direct method. The key points are that

- the Lagrangian is bounded from below, and this gives compactness in the usual way
- the Lagrangian is (strictly) convex wrt p because
  $L_{pp}(x,s,p) = 2(1-\lambda\cos(p^2)) + 4(3+\lambda\sin(p^2))p^2$
  and this is ≥ 0 for every $p \neq 0$ if $\lambda \in (0,1]$. This gives lower semi-continuity and C^1 regularity.

If $\lambda > 1$ the minimum does not exist. Indeed any minimizer $u_0(x)$ satisfies $\dot{u}_0(x_0) = 0$ for some $x_0 \in (0,4)$ (Rolle's theorem), and $L_{pp}(x_0, u_0(x_0), \dot{u}_0(x_0)) = 2(1-\lambda) < 0$. This means that $u_0(x)$ does not satisfy condition (L).

(c) $\boxed{I(\lambda) \to -\infty \text{ as } \lambda \to +\infty}$ It is enough to consider any function $u_0 \in C^1_c((0,4))$ with $u_0 \not\equiv 0$ and $|u_0'(x)|^2 \leq \pi$ for every $x \in [0,4]$ and observe that

$$I(\lambda) \leq F(u_0)$$

and $F(u_0) \to -\infty$ as $\lambda \to +\infty$.

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Remark It can be shown that $I(\lambda) = 4\min\{p^4 + p^2 - \lambda\sin(p^2)\}$, and this is 0 if $\lambda \leq 1$ and negative if $\lambda > 1$.

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4. For every real number $m > 0$, let us set

$$J(m) := \inf\left\{\underbrace{\int_0^1 (u^{19} + \arctan(u^2))\, dx}_{F(u)} : u \in C^1([0,1]),\ u(0) = 0,\ \int_0^1 |\dot u|^7\, dx \le m\right\}.$$

(a) Determine for which values of m it turns out that $J(m)$ is a real number.

(b) Determine whether there exists $m > 0$ such that $J(m) = 0$.

(c) Determine for which real values of α it turns out that

$$\lim_{m\to+\infty} \frac{J(m)}{m^\alpha} = 0.$$

(a) $J(m)$ is a real number $\boxed{\forall m > 0}$

This follow from the direct method. Given a minimizing sequence $\{u_n\}$, the integral constraint and the DBC provide compactness. Therefore

$u_{n_k} \xrightarrow[\text{weakly } L^7]{} v_\infty \qquad u_{n_k} \xrightarrow{\text{unif.}} u_\infty \qquad$ with $v_\infty = \dot u_\infty$.

Now it is enough to observe that u_∞ satisfies the constraints and $F(u_n) \to F(u_\infty)$.

the norm is LSC

Since the min in $W^{1,7}$ is finite, the inf in C^1 is finite as well.

(b) $\boxed{\text{YES}}$ This follows from two facts:

$\rightarrow$ $\exists r > 0 \quad \forall s \in [-r, r] \qquad s^{19} + \arctan(s^2) \ge 0$

$\rightarrow$ $\exists c > 0$ t.c. $|u(x)| \le c\, \|\dot u\|_{L^7} \quad \forall x \in [0,1]$ (usual Hölder regularity of Sobolev functions + DBC)

If m is small enough, then $|u(x)| \le r$ for every $x \in [0,1]$, and hence $F(u) \ge 0$.

(c) The limit is 0 if and only if $\boxed{\alpha < \frac{19}{7}}$. Indeed, with the variable change $u = \sqrt[7]{m}\, v$ we obtain

$$J(m) = m^{\frac{19}{7}} \inf\left\{\underbrace{\int_0^1 v^{19} + \frac{1}{m^{19/7}} \arctan\left(m^{\frac{2}{7}} v^2\right)}_{G_m(v)} : v(0) = 0,\ \int_0^1 |\dot v|^7 \le 1\right\}$$

and hence

$$\frac{J(m)}{m^{19/7}} \to \inf\left\{\int_0^1 v^{19} : v(0) = 0,\ \int_0^1 |\dot v|^7 \le 1\right\} \in (-\infty, 0)\ \}.$$

Gamma convergence or uniform convergence.

easy direct method

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