1. Let us consider the functional

$$F(u) = \int_0^{\pi} \left(\dot{u}^2 - u \sin x\right)\, dx.$$

(a) Discuss the minimum problem for $F(u)$ subject to the condition $\displaystyle\int_0^{\pi} u(x)\, dx = 0$.

(b) Discuss the minimum problem for $F(u)$ subject to the condition $u'(0) = 1$.

(a) In the usual way we find that any minimizer satisfies

$$\begin{cases} 2\ddot{u} = -\sin x + \lambda \\ \dot{u}(0) = 0 \\ \dot{u}(\pi) = 0 \end{cases} \quad \Rightarrow \quad u(x) = \frac{1}{2}\sin x + \lambda x^2 + a + bx$$

Computing λ, a, b using the NBC and the integral constraint we find that the solution is

$$\boxed{u_0(x) = \frac{1}{2}\sin x + \frac{1}{2\pi}x^2 - \frac{1}{2}x + \frac{\pi}{12} - \frac{1}{\pi}}$$

This is the unique min. point because for every $v \in C^1([0,\pi])$ with $\int_0^{\pi} v(x)\, dx = 0$ it turns out that

$$F(u_0 + v) = F(u_0) + \int_0^{\pi} \dot{v}^2 \geq F(u_0)$$

This needs to be checked

with equality $\Leftrightarrow$ $\dot{v} \equiv 0$ $\Leftrightarrow$ $v \equiv$ constant $= 0$ because $\int_0^{\pi} v(x) = 0$.

(b) $\boxed{\inf = -\infty}$ A possible minimizing sequence is

$$u_n(x) = x + n.$$

2. Discuss existence, uniqueness and regularity of the solution to the boundary value problem

$$u'' = u^7 - x^7, \qquad u(0) = 7, \quad u'(7) = 7.$$

Let us consider the Lagrangian

$$L(x,s,p) = \frac{1}{2}p^2 - 7p + \frac{1}{8}s^8 - x^7 s$$

and

$$\min\left\{ \int_0^7 L(x,u,\dot{u})\,dx : u \in H^1((0,7)),\ u(0) = 7 \right\}$$

Any minimizer solves the given equation with extra NBC $L_p = 0$ in $x = 7$, namely $\dot{u}(7) = 7$.

- Existence is a standard application of the direct method. The key point in order to obtain compactness is that there exist $A \in \mathbb{R}$ s.t.

$$L(x,s,p) \geq \frac{1}{2}p^2 - 7p + \frac{1}{8}s^8 - 7^7|s| \geq \frac{1}{4}p^2 - A.$$

- Uniqueness follows in the usual way (which should be made explicit) from two facts
  - → every solution to the equ. is a min point
  - → the min point is unique because of the strict convexity of $L(x,s,p)$ wrt (s,p) for every $x \in [0,7]$.

- Regularity follows in the usual way (again to be made explicit) from ELE in two phases
  - → initial step
  - → bootstrap.

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3. Let us consider, for every $\ell > 0$, the problem

$$\inf\left\{\underbrace{\int_0^\ell \{\ddot{u}^2+\dot{u}^5-\sin(u^2)+u^5\}\,dx}_{F(u)}\,;\ u\in C^1([0,\ell]),\ u(0)=u(\ell)=0\right\}.$$

(a) Determine for which values of ℓ the function $u_0(x)\equiv 0$ is a weak local minimum.

(b) Determine for which values of ℓ the function $u_0(x)\equiv 0$ is a strong local minimum.

(c) Determine for which values of ℓ the infimum is actually a minimum.

(a) u_0 is a WLM $\iff$ $\boxed{\ell\in(0,\pi)}$. The second variation is the quadratic functional

$$Q(v)=\int_0^\ell \dot{v}^2-v^2$$

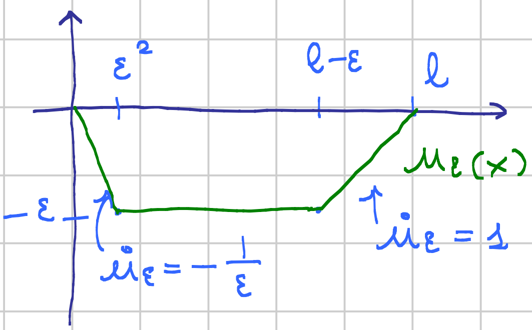
- If $\ell\in(0,\pi)$ then $u_0(x)$ satisfies $(E)+(L^+)+(J^+) \Rightarrow$ WLM
- If $\ell>\pi$ then $u_0(x)$ satisfies (L^+) but not $(J) \Rightarrow$ NO WLM
- If $\ell=\pi$ we observe that

$$F(-\varepsilon\sin x)=\int_0^\pi \underbrace{\varepsilon^2\cos^2x-\varepsilon^2\sin^2x}_{\text{equal}}-\varepsilon^5\sin^5x-\underset{0}{\underset{\shortparallel}{\varepsilon^5\cos^5x}}+O(x^5)<0$$

when ε is small enough.

(b) u_0 is $\boxed{\text{NEVER}}$ a SLM because it does not satisfy (W)

(c) The infimum is $\boxed{\text{NEVER}}$ a minimum, and actually it is equal to $-\infty$ for every $\ell>0$.
A possible minimizing sequence is the following



A standard computation shows that $F(u_\varepsilon)\to-\infty$.

Remark
- u_ε as in (c) shows also that $u_0(x)$ is not a SLM
- the relaxation of $F(u)$ is $\equiv-\infty$, because the convexification of p^2+p^5 is $\equiv-\infty$.

4. Let us set, for every $\varepsilon > 0$,

$$m_\varepsilon := \inf \left\{ \int_0^4 \left(\dot{u}^2 - u \sin(u^2) \right) dx : u \in C^1([0,4]),\ u(0) = 0,\ u(4) = \varepsilon \right\}.$$

(a) Determine for which values of ε the infimum is actually a minimum.

(b) Compute the leading term of m_ε as $\varepsilon \to 0^+$.

(c) Compute the leading term of m_ε as $\varepsilon \to +\infty$.

(a) The infimum is a min $\boxed{\forall \varepsilon > 0}$ This follows from the direct method. The key point is that

$$F(u) \geq \int_0^4 \dot{u}^2 - |u|$$

Now as usual we can estimate $|u(x)|$ with $\|\dot{u}\|_{L^2}$.

(b) $\boxed{m_\varepsilon \sim \frac{1}{4}\varepsilon^2 \text{ as } \varepsilon \to 0^+}$ With the variable change $u = \varepsilon v$ we obtain

$$m_\varepsilon = \varepsilon^2 \inf \Big\{ \underbrace{\int_0^4 \dot{v}^2 - \frac{1}{\varepsilon} v \sin(\varepsilon^2 v)}_{G_\varepsilon(v)} : v(0) = 0,\ v(1) = 1 \Big\}$$

$$\frac{m_\varepsilon}{\varepsilon^2} \to \inf \left\{ \int_0^4 \dot{v}^2 : v(0) = 0,\ v(1) = 1 \right\} \underset{\uparrow}{=} \frac{1}{4}$$

trivial indirect method

The key point is that $G_\varepsilon(v) \xrightarrow{\Gamma} \int_0^4 \dot{v}^2$, which in turns follows from the estimate

$$p^2 - \frac{1}{\varepsilon} s \sin(\varepsilon^2 s) \geq p^2 - \varepsilon s^2$$

(c) $\boxed{m_\varepsilon \sim \frac{1}{4}\varepsilon^2 \text{ as } \varepsilon \to +\infty}$ The argument is the same as before.

The key point is that

$$G_\varepsilon(v) \xrightarrow{\Gamma} \int_0^4 \dot{v}^2 \qquad \text{also as } \varepsilon \to +\infty.$$

This follows in a standard way from the inequality

$$p^2 - \frac{1}{\varepsilon} s \sin(\varepsilon^2 s) \geq p^2 - \frac{1}{\varepsilon} |s|.$$

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